Fermions without Vierbeins

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October 5, 2012
From Flat to Curved Space
aim: describe fermions in curved spacetime $\Rightarrow$ need $\gamma$ matrices

Clifford Algebra

flat space: $\{\gamma_a, \gamma_b\} = 2\eta_{ab}I$

$\Rightarrow$ simplest ansatz in curved space: $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}I + \alpha R_{\mu\nu}I$?
Clifford Algebra

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Clifford Algebra

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⇒ simplest ansatz in curved space: $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}I + \alpha R_{\mu\nu}I$?
other motivation:
What leads to the $\gamma$ matrices and the Clifford algebra, actually?

Dirac Equation: $\slashed{D}\psi - m\psi = 0$

Klein-Gordon Equation: $\Box \psi - m^2 \psi = 0$
simplest covariant formulation:

- replace $\partial_a \rightarrow \nabla_\mu$ and $\gamma_a \rightarrow \gamma_\mu$
- add a curvature term, but in a “simple” manner

Dirac Equation: $\nabla \psi - m\psi = 0$

Klein-Gordon Equation: $\nabla^2 \psi + f(R_{\mu\nu\rho\lambda})\psi - m^2 \psi = 0$
from these equations we get the Clifford algebra

\[ \{ \gamma^\mu(x), \gamma^\nu(x) \} = 2g^{\mu\nu}(x)I \]

but we get two more important conclusions

\[ \frac{1}{4} [\gamma^\mu(x), \gamma^\nu(x)] [\nabla_\mu, \nabla_\nu] \psi(x) = f(R_{\mu\nu\rho\lambda}(x)) \psi(x) \]

\[ \gamma^\mu(x) (\nabla_\mu \gamma^\nu(x)) = 0 \]
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Problems

I) How do we get the $\gamma$ matrices?

II) How do we get the covariant derivative?

$\Rightarrow$ Which symmetries should it respect?
Vierbein Formalism
flat space Clifford algebra is well known

⇒ define the Vierbein: $e_\mu^\ a(x)$ via local inertial coordinates

with:
1. $\eta^{ab} = e_\mu^\ a(x)e_\nu^\ b(x)g^{\mu\nu}(x)$
2. $g_{\mu\nu}(x) = e_\mu^\ a(x)e_\nu^\ b(x)\eta_{ab}$

⇒ define the $\gamma$ matrices: $\gamma_\mu(x) = e_\mu^\ a(x)\gamma_a$ via the Vierbein$^1$

$^1$[DeWitt]
Symmetries

Which symmetries do we have?

- coordinate transformations:
  \[ e'_{\mu} \cdot a = \frac{\partial x^\nu}{\partial x'^\mu} e_\nu \cdot a \]
  \[ \gamma'_{\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \gamma_\nu \]
  \[ \psi' = \psi \]

- local Poincaré-transformations:
  \[ \tilde{e}_{\mu} \cdot a = \Lambda^a_{\ b} e_{\mu} \cdot b \]
  \[ \tilde{\gamma}_{\mu} = \tilde{e}_{\mu} \cdot a \gamma_a = S_P(\Lambda) \gamma_{\mu} S_P^{-1}(\Lambda) \]
  \[ \tilde{\psi} = S_P(\Lambda) \psi \]
need to choose special coordinate system

\[
\gamma_*(x) = \frac{i(-i)^{d/2}}{d!} \sqrt{-g(x)} \varepsilon_{\mu_1...\mu_d} \gamma^{\mu_1}(x) \ldots \gamma^{\mu_d}(x)
\]

\[
\equiv i(-i)^{d/2} \gamma^{(0)} \ldots \gamma^{(d-1)} = \gamma(^*)
\]

should respect the full symmetry of the Clifford algebra

\[S(x) \in GL(d_\gamma, \mathbb{C})\] (spinbase transformations)
Weldon Formalism

Advanced Symmetry and Covariant Derivative

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Fermions without Vierbeins
pretend we have a set of $\gamma$ matrices, which satisfies:

$$\{\gamma_\mu(x), \gamma_\nu(x)\} = 2g_{\mu\nu}(x)I$$

Transformations:
- coordinate transformations
- spinbase transformations
Spin Metric

need a “covariant” spinor $\tilde{\psi}$, with $\tilde{\psi} = \psi S^{-1}$

$\Rightarrow$ construct $\tilde{\psi}$ using the hermitean conjugated spinor $\psi^\dagger$

introduce spin metric $h$, with $\tilde{h} = S^{-1\dagger} h S^{-1}$

define Dirac conjugated spinor:

$$\bar{\psi} = \psi^\dagger h \quad \text{(in flat space: } \bar{\psi} = \psi^\dagger \gamma^{(0)})$$

Spin Metric

$$\gamma^{\mu\dagger} = -h \gamma^{\mu} h^{-1}, \quad h^\dagger = -h$$

compare to flat space: $h = \gamma^{(0)}$
Covariant Derivative

What are the concepts of covariant differentiation? ⇒ look at GR

(i) linearity: \[ D_\mu (T_1^\nu + T_2^\nu) = D_\mu T_1^\nu + D_\mu T_2^\nu \]

(ii) product rule: \[ D_\mu (T_1^\nu T_2^\rho) = (D_\mu T_1^\nu) T_2^\rho + T_1^\nu (D_\mu T_2^\rho) \]

(iii) index structure: \[ D_\mu T_\nu = (D_\mu T^\rho) g_{\rho\nu} \]

(iv) no torsion: \[ D_\mu T_\nu - D_\nu T_\mu = \partial_\mu T_\nu - \partial_\nu T_\mu, \]
What are the concepts of covariant differentiation? ⇒ look at GR

⇒ adopt these rules

(i) linearity: 
\[ (\nabla_\mu (\psi_1 + \psi_2))^I = (\nabla_\mu \psi_1)^I + (\nabla_\mu \psi_2)^I \]

(ii) product rule: 
\[ (\nabla_\mu (\psi \bar{\psi}))^I_J = (\nabla_\mu \psi)^I \bar{\psi}_J + \psi^I (\nabla_\mu \bar{\psi})_J \]

(iii) index structure: 
\[ (\nabla_\mu \bar{\psi})_I = (\bar{\nabla}_\mu \psi)_I \]

(iv) DE + KGE: 
\[ \gamma^\mu \nabla_\mu \gamma^\nu = 0 \]

(v) hermiticity: 
\[ (\nabla_\mu \psi^\dagger)^I = (\nabla_\mu \psi)^\dagger_I \]
for example we get

\[ \nabla_\mu \psi = D_\mu \psi + \Gamma_\mu \psi \]

\[ \nabla_\mu \gamma^\nu = D_\mu \gamma^\nu + [\Gamma_\mu, \gamma^\nu] \]

\( \Gamma_\mu \): affine spinor connection

\( D_\mu \): covariant derivative for spacetime indices
Weldon Theorem
still need the affine connection ⇒ look at $\gamma^\mu(\nabla_\mu \gamma^\nu) = 0$

think about possible variations of the $\gamma^\mu$ in the Clifford algebra

⇒ Weldon Theorem$^2$

Weldon Theorem

$$\Delta \gamma^\mu = \frac{1}{2} (\Delta g^{\mu\nu}) \gamma^\nu - [M, \gamma^\mu], \quad \text{tr } M = 0$$

bijective: $\Delta \gamma^\mu \leftrightarrow \Delta g^{\mu\nu}, M$

$^2$[Weldon]
Affine Connection

the theorem implies: \( \nabla_\mu \gamma^\nu = D_\mu \gamma^\nu + [\Gamma_\mu, \gamma^\nu] = 0 \)

\[ \Rightarrow \] reconstruct the affine spinor connection \( \Gamma_\mu \) from the \( \gamma^\nu \)

for example in \( d = 4 \):

\[
\Gamma_\mu = t_\mu \cdot \alpha^\beta [\gamma_\alpha, \gamma_\beta] + v_\mu \cdot \alpha \gamma_\alpha + a_\mu \cdot \alpha \gamma_\alpha \gamma_* + p_\mu \gamma_* + i q A_\mu I
\]

\[
t_\mu \cdot \alpha^\beta = -\frac{1}{32} \text{tr} (\gamma^\alpha D_\mu \gamma^\beta)
\]

\[
v_\mu \cdot \alpha = \frac{1}{48} \text{tr} ([\gamma^\alpha, \gamma_\nu] D_\mu \gamma^\nu)
\]

\[
a_\mu \cdot \alpha = \frac{1}{8} \text{tr} (\gamma_* \partial_\mu \gamma^\alpha)
\]

\[
p_\mu = \frac{1}{32} \text{tr} (\gamma_* \gamma_\nu \partial_\mu \gamma^\nu)
\]
Vierbein vs. Weldon

Vierbein Formalism: just a linear combination of flat $\gamma$ matrices
\[
\gamma_\mu(x) = e_\mu^a(x)\gamma_a
\]

Weldon Formalism: full Clifford algebra base
\[
\gamma_\mu(x) = T_\mu^{ab}(x)[\gamma_a, \gamma_b] + V_\mu^a(x)\gamma_a + A_\mu^a(x)\gamma_a\gamma(\ast) + P_\mu(x)\gamma(\ast)
\]
(for example in $d = 4$)

$\Rightarrow$ Vierbein Formalism is like special gauge in electrodynamics
\(\gamma^\mu\) appear to be the fundamental degree of freedom

define field strength like in gauge theory: \(\Phi_{\mu\nu}\psi = [\nabla_\mu, \nabla_\nu]\psi\)

\[
\Rightarrow \Phi_{\mu\nu} = \frac{1}{8} R_{\mu\nu\alpha\beta}[\gamma^\alpha, \gamma^\beta] + iqF_{\mu\nu}I, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\]

can now construct action \textbf{linear} in the field strength:

\[
\Rightarrow S[\gamma^\mu] = -\frac{1}{d_\gamma} \int dx \sqrt{-g(x)} \text{ tr } (\Phi_{\mu\nu}(x)[\gamma^\mu(x), \gamma^\nu(x)])
\]

\[
\equiv \int dx \sqrt{-g(x)} R(x) \text{ (Einstein-Hilbert action)}
\]
\( \gamma^\mu \) appear to be the fundamental degree of freedom

define field strength like in gauge theory: \( \Phi_{\mu\nu} \psi = [\nabla_\mu, \nabla_\nu] \psi \)

\[ \Rightarrow \Phi_{\mu\nu} = \frac{1}{8} R_{\mu\nu\alpha\beta} [\gamma^\alpha, \gamma^\beta] + iq F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

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\[ \Rightarrow S[\gamma^\mu] = -\frac{1}{d_\gamma} \int dx \sqrt{-g(x)} \ \text{tr} \ (\Phi_{\mu\nu}(x)[\gamma^\mu(x), \gamma^\nu(x)]) \]
\[ \equiv \int dx \sqrt{-g(x)} \ R(x) \ (\text{Einstein-Hilbert action}) \]
Summary:

- respect full symmetry of Clifford algebra  
  ⇒ spinbase transformations
- construct the spin metric $h$ from the $\gamma^\mu$
- construct the affine spinor connection $\Gamma_\mu$ from the $\gamma^\mu$

Outlook:

- do QFT in curved space, without Vierbeins
- more natural formulation for fermion systems on curved space
- application to Quantum Gravity?
References
