Worm Algorithms for the QCD Phase Diagram with Effective Theories

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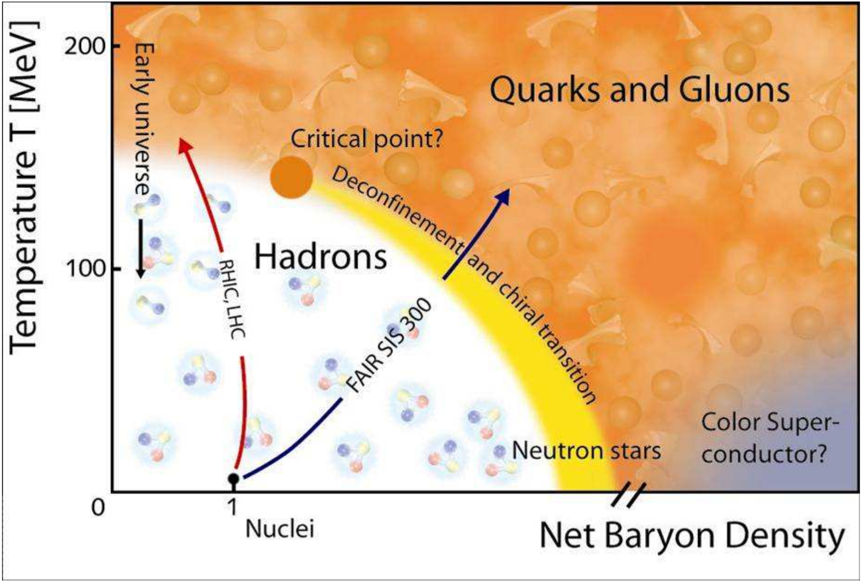
FWF Der Wissenschaftsfonds.

Something we would like to understand



From the continuum to the lattice

- Gluon and quark actions in the continuum

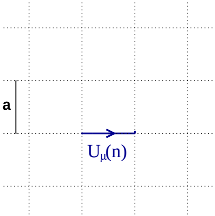
$$S_F = \sum_{f=1}^{N_f} \int d^4x \, \overline{\psi_f}(x) \left[\gamma_\nu (\partial_\nu + i A_\nu(x)) + m_f \right] \psi_f(x)$$

$$S_G = \frac{1}{2g^2} \int d^4x \, Tr[F_{\mu\nu}(x) F_{\mu\nu}(x)]$$

- Discretizing space and time

$$\int d^4x \rightarrow a^4 \sum_x$$

$$\partial_\nu \psi(x) \rightarrow \frac{1}{2a} \left[\psi(x + a\hat{\nu}) - \psi(x - a\hat{\nu}) \right]$$



- The gluonic fields:

$$A_\nu(x) \rightarrow U_\nu(x) = e^{iaA_\nu(x)} \in SU(3)$$

QCD on the lattice

- Partition sum

$$Z = \int D[\psi, \overline{\psi}, U] e^{-S_G - S_F} = \int D[U] e^{-S_G[U]} det D[U]^{N_f}$$

- Monte Carlo simulations: Generate gauge configurations with probability

$$P[U] \propto e^{-S_G[U]} det D[U]^{N_f}$$

- Vacuum expectation value of observables

$$\langle O \rangle = \frac{1}{Z} \int D[U] e^{-S_G[U]} det D[U]^{N_f} O[U]$$

Lattice QCD with chemical potential: sign problem

- Introducing the chemical potential μ
  - Continuum:

$$S_F = \sum_{f=1}^{N_f} \int d^4x \, \overline{\psi}_f(x) \big[\gamma_\nu(\partial_\nu + iA_\nu(x)) + m_f + i\mu\gamma_0\big] \psi_f(x)$$

  - Lattice:

$$Z = \int D[q,\overline{q},A] e^{-S_G[A] - S_F[q,\overline{q},A;\mu]}$$
$$= \int D[A] e^{-S_G[A]} (detD[A;\mu])^{N_f}$$

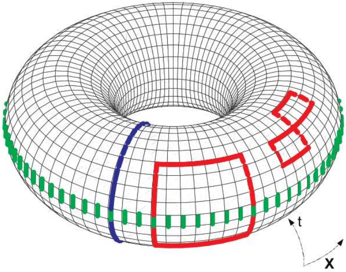
- $detD[A;\mu]$ is complex for $\mu > 0$
- cannot be used as a probability in a Monte Carlo simulation

- We explore new ideas in effective theories

Deconfinement

- Observable for a single static quark:
  Polyakov loop

$$P(\vec{x}) = \mathcal{P}\, exp\big(i\int_0^\beta dt A_0(t,\vec{x})\big) \rightarrow \prod_{t=1}^{\beta} U_0(t,\vec{x})$$



- determines the free energy of a single quark F_q

$$\langle Tr\ P\rangle \propto e^{-F_q/T}$$

- Then, $\langle Tr\ P\rangle$ is an order parameter for deconfinement
  - $T < T_c$: $\langle Tr\ P\rangle = 0$: quarks are confined
  - $T > T_c$: $\langle Tr\ P\rangle \neq 0$: quarks are deconfined

Center symmetry and deconfinement

- Center transformation

  $z \in \mathbb{Z}_3 \; = \; \{\mathbb{1}, e^{i2\pi/3}\mathbb{1}, e^{-i2\pi/3}\mathbb{1}\}$: $U_0(t^*, \vec{x}) \; \rightarrow \; zU_0(t^*, \vec{x})$

- The path integral measure and the gauge action are invariant

- The Polyakov loop is transformed by an element of the center

$$P \rightarrow zP \quad , \; z \in \mathbb{Z}_3$$

- Then, $\langle Tr\ P \rangle$ is an order parameter for the center symmetry
  - $T < T_c$: $\langle Tr\ P \rangle = 0$: unbroken symmetry $\rightarrow$ quarks are confined
  - $T > T_c$: $\langle Tr\ P \rangle \neq 0$: broken symmetry $\rightarrow$ quarks are deconfined

Effective gauge action - Yaffe and Svetitsky conjecture

- Taking the strong coupling approximation of $S_G[U] \rightarrow$ effective action:

$$S_{eff} = \tau \sum_{x} \sum_{\nu=1}^{3} \left[TrP(x)\, TrP(x+\hat{\nu})^{\dagger} + h.c. \right]$$

  - $\tau = \tau(T)$ increasing function of T

- **Yaffe and Svetitsky conjecture:**
  Deconfinement transition of $SU(3)$ in $4d$ can be described by an effective $3d$ spin system $\in \mathbb{Z}_3$.

$$TrP(x) \rightarrow p(x) \;\in\; \{1, e^{+i2\pi/3}, e^{-i2\pi/3}\}$$

and:

$$\int_{SU(3)} D[P] \rightarrow \sum_{p(x) \in \mathbb{Z}_3}$$

Yaffe and Svetitsky (1981)

Adding fermions:

- The fermionic action with chemical potential breaks center symmetry
- Add a center symmetry breaking term in the effective theory

$$\sum_x \kappa \left[e^{\mu} TrP(x) \, + \, e^{-\mu} TrP(x)^{\dagger} \right]$$

  - $\kappa = \kappa(m_q)$ decreasing function of m_q
  - μ is the chemical potential

An effective theory for QCD thermodynamics

$$S_{eff} = -\sum_{x}\left(\tau\sum_{\nu=1}^{3}\left[L(x)L(x+\hat{\nu})^{\dagger}+h.c.\right] + \kappa\left[e^{\mu}L(x) + e^{-\mu}L(x)^{\dagger}\right]\right)$$

- Gauge action $\rightarrow$ nearest neighbor interaction for $L(x) = Tr\ P(x)\ \in SU(3)$.
  Temperature $\rightarrow \tau$ (increases with T).
  Spontaneous center symmetry breaking $\rightsquigarrow$ deconfinement transition.
- Fermion determinant (hopping expansion) $\rightarrow$ magnetic term.
  Quark mass $\rightarrow \kappa$ (decreases with m_q).
  Breaks center symmetry. Complex phase problem.

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  Breaks center symmetry. Complex phase problem.
- We will study the flux representation of two models:
  - $\mathbb{Z}_3$ model: $TrP(x) \rightarrow p(x) \in \mathbb{Z}_3$
    Y.D., H. G. Evertz, C. Gattringer, Phys. Rev. Lett. **106** (2011) 222001
  - $SU(3)$ model
    C. Gattringer, Nucl. Phys. B **850** (2011) 242

$\mathbb{Z}_3$: Flux representation - 1

- Effective center model still has complex action $\Rightarrow$ new variables!
- For the neighbor interaction and magnetic term, we use the Ansatz:

$$e^{\tau[p(x)p(x+\hat{\nu})^* + c.c.]} = C \sum_{b_{x,\nu}=-1}^{+1} B^{|b_{x,\nu}|} \left(p(x)p(x+\hat{\nu})^* \right)^{b_{x,\nu}}$$

$$e^{\kappa e^{\mu} p(x) + \kappa e^{-\mu} p(x)^*} = \sum_{s_x=-1}^{+1} M_{s_x} \, p(x)^{s_x}$$

- New variables:
  - **dimers** : $b_{x,\nu} \in \{-1, 0, +1\}$ on the link (x, ν).
  - **monomers**: $s_x \in \{-1, 0, +1\}$ on the site x.

$\mathbb{Z}_3$: Flux representation - 2

- The partition function in the flux representation:

$$Z = \sum_{\{b,s\}} \left(\prod_{x,\nu} B^{|b_{x,\nu}|} \right) \left(\prod_{x} M_{s_x} \right) \prod_{x} T \left(\sum_{\nu} [b_{x,\nu} - b_{x-\hat{\nu},\nu}] + s_x \right)$$

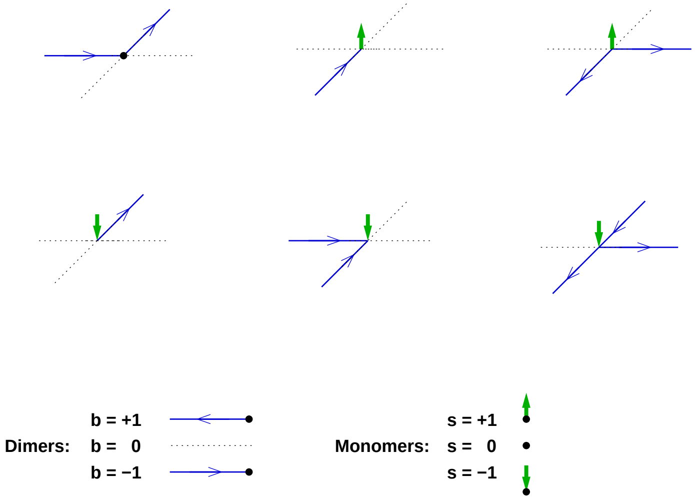
- Only real and positive contributions.

- Constraint $T(x)$: flux conservation modulo 3 at every site.

F. Karsch et al.(1984)

A. Patel,T. DeGrand,C. DeTar(1983)

$\mathbb{Z}_3$: Graphical representation of admissible terms

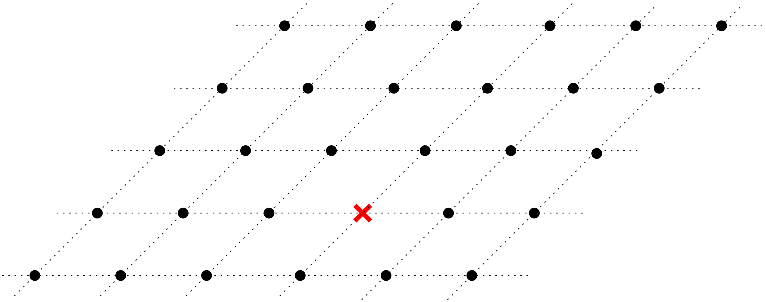


Numerical analysis

- Study the $\tau - \mu$ phase diagram.
- Identify the phase boundaries between confinement and deconfinement.
- Analyze the nature of the transitions.
- Location of the transition lines determined from the maxima of the Polyakov loop susceptibility χ_P and the heat capacity C.
- MC simulation: Worm algorithm of the flux representation.
  N. Prokof'ev and B. Svistunov (2001)

Worm Algorithm:

1. The worm starts at a random position of the lattice.



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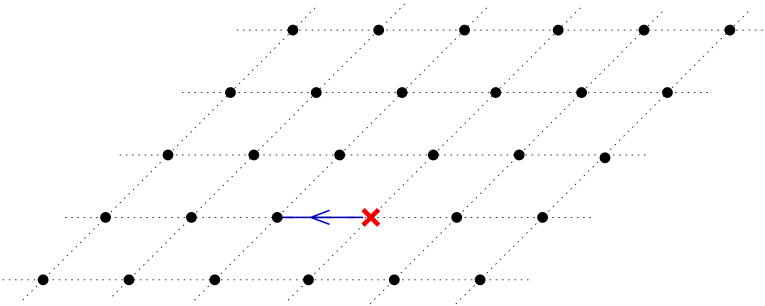
2. It may decide to insert dimers or monomers. Each update is accepted or rejected using the Metropolis criterion.



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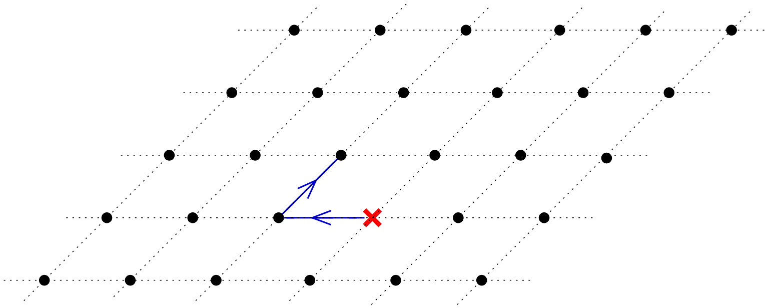

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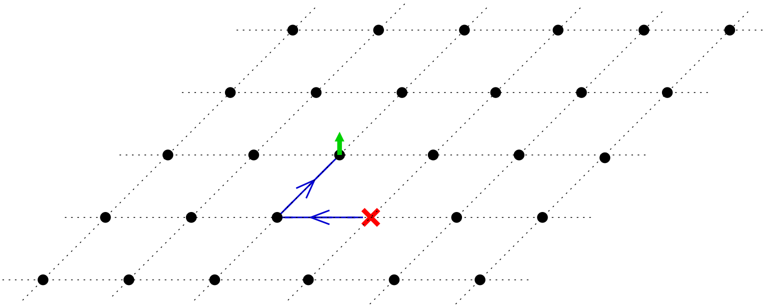

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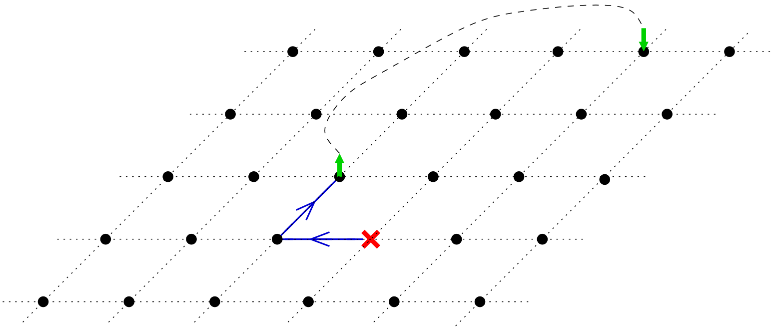

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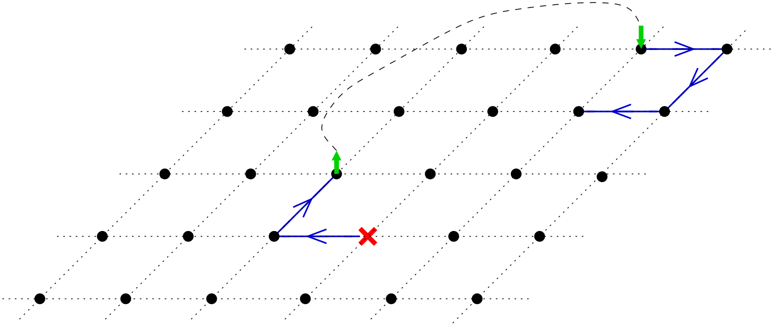
4. These steps are continued until the worm closes.



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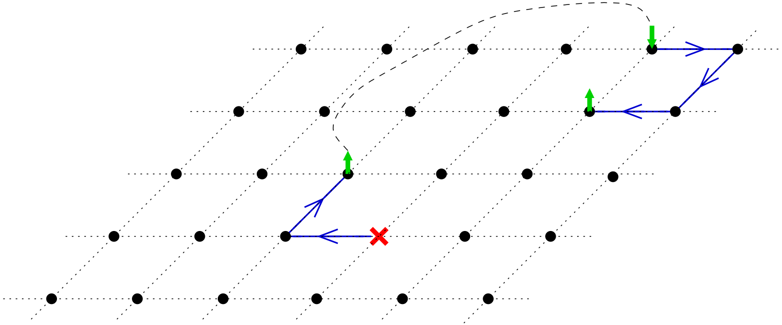

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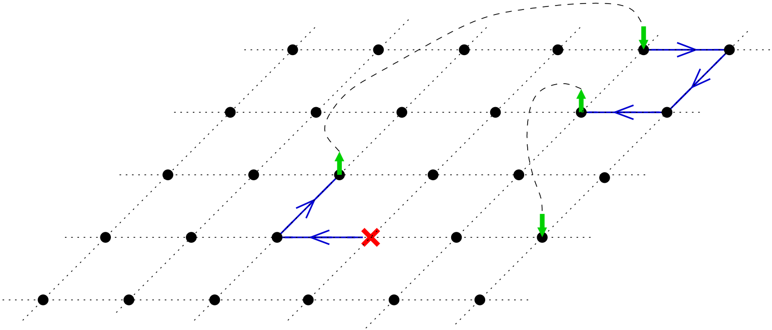

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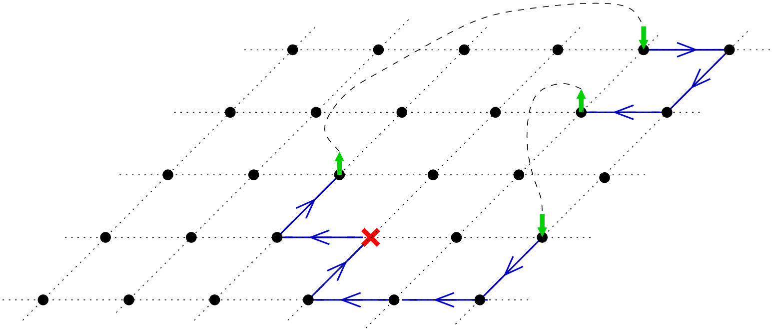
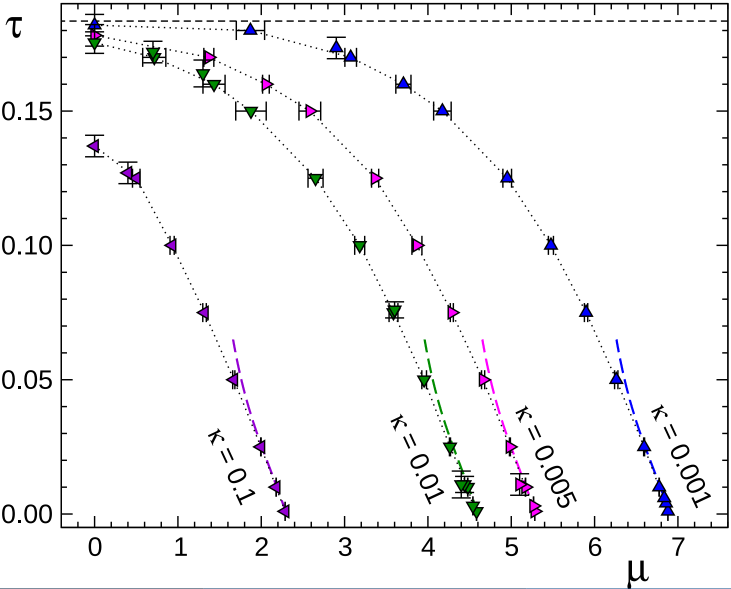


$\mathbb{Z}_3$: Phase diagram from χ_P



Flux representation for $SU(3)$ model

- For the neighbor interaction term:

$$e^{\tau[L(x)L(x+\hat{\nu})^\dagger + L(x)^\dagger L(x+\hat{\nu})]} =$$

$$\sum_{l_{x,\nu}=0}^{+\infty} \frac{\tau^{l_{x,\nu}}}{l_{x,\nu}!} \left(L(x)L(x+\hat{\nu})^\dagger\right)^{l_{x,\nu}} \sum_{\bar{l}_{x,\nu}=0}^{+\infty} \frac{\tau^{\bar{l}_{x,\nu}}}{\bar{l}_{x,\nu}!} \left(L(x)^\dagger L(x+\hat{\nu})\right)^{\bar{l}_{x,\nu}}$$

- The magnetic term:

$$e^{\kappa e^{\mu} L(x) + \kappa e^{-\mu} L(x)^\dagger} =$$

$$\sum_{s_x=0}^{+\infty} \frac{(\kappa e^{\mu})^{s_x}}{s_x!} L(x)^{s_x} \sum_{\bar{s}_x=0}^{+\infty} \frac{(\kappa e^{-\mu})^{\bar{s}_x}}{\bar{s}_x!} \left(L(x)^\dagger\right)^{\bar{s}_x}$$

- New variables:
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$SU(3)$: Flux representation

- The partition function in the flux representation:

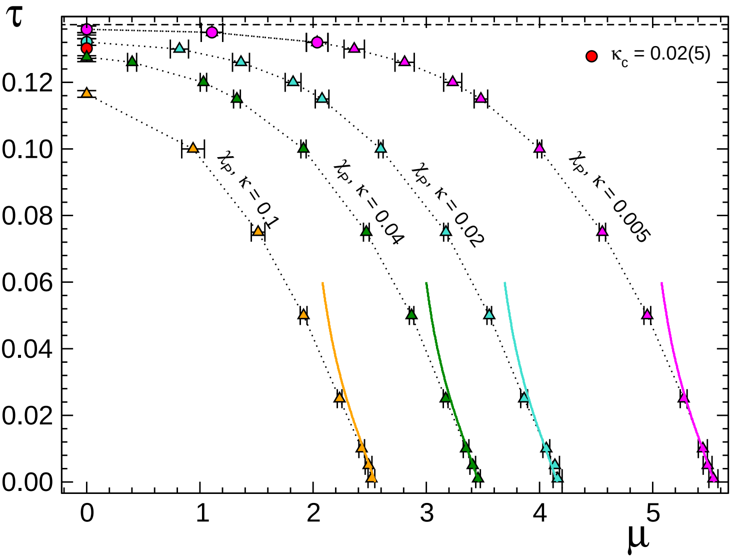
$$Z = \sum_{\{l,\bar{l}\}} \sum_{\{s,\bar{s}\}} \left(\prod_{x,\nu} \frac{\tau^{l_{x,\nu}+\bar{l}_{x,\nu}}}{l_{x,\nu}!\bar{l}_{x,\nu}!} \right) \left(\prod_{x} \frac{(\kappa e^{\mu})^{s_x} (\kappa e^{-\mu})^{\bar{s}_x}}{s_x!\bar{s}_x!} \right) \times \left(\prod_{x} \int D[P] (TrP(x))^{f(x)} (TrP(x)^{\dagger})^{\bar{f}(x)} \right)$$

  - $f(x) \;=\; \sum_{\nu=1}^{3} [l_{x,\nu} + \bar{l}_{x-\hat{\nu},\nu}] \;+\; s_x$
  - $\bar{f}(x) \;=\; \sum_{\nu=1}^{3} [l_{x-\hat{\nu},\nu} + \bar{l}_{x,\nu}] \;+\; \bar{s}_x$

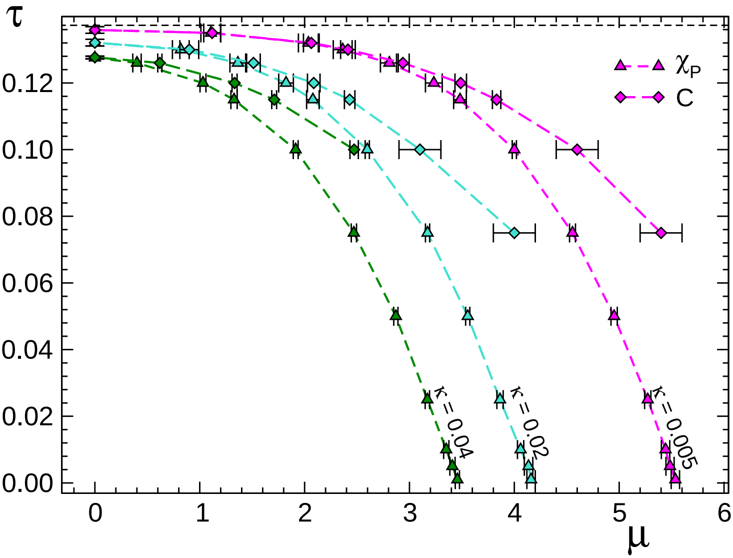
- The $SU(3)$ integrals:

$$\int dP (TrP)^{f} (TrP^{\dagger})^{\bar{f}} \;=\; I(f|\bar{f}) \;=\; \begin{cases} I_{f\bar{f}} & (f-\bar{f}) \; mod \; 3 = 0 \\ 0 & else \end{cases}$$

$SU(3)$: Phase diagram from χ_P



Comparison of phase boundaries from C and χ_P



Summary

- We have studied an effective center theory of QCD with finite quark density at non-zero temperature.
- In the flux representation the model is free of the complex phase problem and can be simulated with a generalized worm algorithm.
- When only center degrees of freedom are considered all transitions are of crossover type (unless $m_q >> 1$).
- We generated reference results at finite μ which can be used to test other approaches.
- Ongoing work:
  - Implementation of $SU(3)$ model using the worm algorithm.
  - Determination of critical line.

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Thank you for your attention!