The Schwinger Model in Point Form

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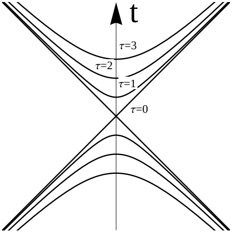
Point form

Point form of relativistic dynamics †:

- Quantization surfaces: Spacetime hyperboloids

$$\boxed{x^2} \equiv x^\mu x_\mu \equiv t^2 - \vec{x}^2 = \boxed{\tau^2} = const.$$



  - Spacelike surfaces with surface element

$$\boxed{d\sigma_\mu = 2\, d^4x\, \delta(x^2 - \tau^2)\, \theta(x^0)\, x_\mu}$$

  - Invariant under action of Lorentz group
  - Coordinate vector x^μ is normal vector

- Dynamic (interaction-dependent) Poincaré generators: P^μ
- Kinematic generators: $\{K^i, J^k\} \sim J^{\mu\nu} \Rightarrow$ Lorentz group!

(Comp. w/ instant form: Equal-time surfaces; dynamic: H, K^i; kinematic: J^k)

† P. A. M. Dirac: "Forms of Relativistic Dynamics", Rev. Mod. Phys. 1949, 21

Schwinger Model

The Schwinger Model[†]:

QED of massless fermions in 1+1 dimensions with Lagrangian

$$\boxed{\begin{aligned} \mathcal{L} = \quad & \mathcal{L}_e \quad + \quad \mathcal{L}_f \quad + \quad \mathcal{L}_{int} \quad = \\ = & -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + \tfrac{i}{2} \overline{\psi} \, \partial\!\!\!/ \, \psi + \tfrac{1}{2} e \overline{\psi} \, A\!\!\!/ \, \psi + \text{h.c.} \end{aligned}}$$

with Dirac matrices (Weyl basis) and metric tensor

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

[†] J. S. Schwinger: "Gauge Invariance and Mass 2", Phys. Rev. 1962, 128

Motivation: Schwinger model

Why study the Schwinger model in point form?

Answer: The Schwinger model ...

- is exactly solvable for massless fermions
- is superrenormalizable (e has dimension of mass)
- leads to phenomena one would like to study in other, 4-dimensional theories (e.g. confinement in QCD)
- has so far been solved ...
  - perturbatively
  - using functional methods
  - using the operator approach in instant form and in front form
- would be interesting to study also in point form
- is a tool to test that approach against other methods

Motivation: Point form

On the other hand, the point form of relativistic dynamics . . .

- has been useful in the study of few-body systems
- exhibits intrinsic Lorentz covariance
- has not been studied much in local field theories
- may lead to new approaches in non-perturbative study of 4-dimensional field theories

Contents

Lagrangian

Fermionic part

$$\mathcal{L}_f = \frac{i}{2}\left(\overline{\psi}\,\gamma^\mu \partial_\mu\,\psi\right)$$

Photonic part

$$\partial_\mu A^\mu = 0 \quad \Rightarrow \quad \mathcal{L}_e = -\frac{1}{2}\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu\right)$$

(Lorenz gauge ⇒ Gupta–Bleuler quantization)

Energy–momentum tensor

$$\boxed{\mathcal{L}_f = \frac{i}{2}\left(\overline{\psi}\,\gamma^\mu \partial_\mu\,\psi\right) \qquad \mathcal{L}_e = -\frac{1}{2}\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu\right)} \quad \Rightarrow$$

Fermionic part

$$\Theta_f^{\mu\nu} = \left(\partial^\nu \overline{\psi}\right)\frac{\partial \mathcal{L}_f}{\partial\left(\partial_\mu \overline{\psi}\right)} + \frac{\partial \mathcal{L}_f}{\partial\left(\partial_\mu \psi\right)}\left(\partial^\nu \psi\right) - g^{\mu\nu}\mathcal{L}_f =$$

$$= \frac{i}{2}\,\overline{\psi}\,\gamma^\mu \overleftrightarrow{\partial}^\nu\,\psi$$

Photonic part

$$\Theta_e^{\mu\nu} = \frac{\partial \mathcal{L}_e}{\partial\left(\partial_\mu A^\rho\right)}\,\partial^\nu A^\rho - g^{\mu\nu}\mathcal{L}_e =$$

$$= -(\partial^\mu A_\rho)(\partial^\nu A^\rho) + \frac{1}{2}\,g^{\mu\nu}\,(\partial_\lambda A^\rho)(\partial^\lambda A_\rho)$$

2-Momentum operator

via point–form integration of energy-momentum tensor:

$$P^\mu = i \int_{\mathbb{R}^2} \underbrace{d^2x\, \delta\left(x^2 - \tau^2\right)\, \theta(x^0)\, x_\nu}_{d\sigma_\nu} \Theta^{\nu\mu}(x)$$

Decomposition into plane waves

Fermionic part

$$\psi(x) = \int \frac{dp^1}{2p^0\sqrt{2\pi}} \Big(c(p^1)u(p^1)e^{-ixp} + d^\dagger(p^1)v(p^1)e^{ixp} \Big)$$

(on-shell integration: $p^2 = m^2$;
$c^{(\dagger)}$, $d^{(\dagger)}$ ... creation/ annihilation ops., u, v ... basis spinors)

Photonic part

$$A^\mu(x) = \int \frac{dk^1}{2k^0\sqrt{2\pi}} \sum_{\alpha=0}^{1} \Big(a^\mu_\alpha(k^1)\, e^{-ikx} + a^{\dagger\mu}_\alpha(k^1)\, e^{ikx} \Big)$$

($k^2 = 0$; $a^{(\dagger)\mu}_\alpha = a^{(\dagger)}_\alpha \epsilon^{(*)\mu}_\alpha$ ... creation/ annihil. ops. $\times$ pol. vectors)

Momentum representation

Insert

$$\psi = \int_p \left(c_p u_p e^{-ixp} + d_p^\dagger v_p e^{ixp} \right)$$

and

$$A^\mu = \int_k \sum_\alpha \left(a_\alpha^\mu(k)\, e^{-ikx} + a_\alpha^{\dagger\mu}(k)\, e^{ikx} \right)$$

into

$$\Theta_f^{\mu\nu} = \frac{i}{2} \overline{\psi} \gamma^\mu \overleftrightarrow{\partial}^\nu \psi$$

and

$$\Theta_e^{\mu\nu} = -\partial^\mu A_\rho \partial^\nu A^\rho + \frac{1}{2} g^{\mu\nu} \partial_\lambda A^\rho \partial^\lambda A_\rho$$

and then into

$$P^\mu = i \int d\sigma_\nu \Theta^{\nu\mu}$$

$\Rightarrow$

Fermionic part

$$P_f^\mu = 2 \int_{\mathbb{R}^2} d\sigma_\nu \iint_{p,p'} \sum (p \pm p')^\mu \left[\begin{pmatrix} c \\ d \end{pmatrix}_p^{(\dagger)} \begin{pmatrix} c \\ d \end{pmatrix}_{p'}^{(\dagger)} \right] \left[\begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}_p \gamma^\nu \begin{pmatrix} u \\ v \end{pmatrix}_{p'} \right] e^{\pm i(p \pm p')x}$$

$\longleftarrow$ (bilinear combinations) $\longrightarrow$

Photonic part

$$P_e^\mu = 2 \int_{\mathbb{R}^2} d\sigma_\nu \iint_{k,k'} \underset{(\text{tensor} \sim k'k)}{T^{\mu\nu}(k,k')} \sum_{\alpha,\beta,\dagger} \pm \left(a_\alpha^{(\dagger)}(k) \cdot a_\beta^{(\dagger)}(k') \right) e^{\pm i(k \pm k')x}$$

Covariant distribution W^μ

$$P_f^\mu = 2\int_{\mathbb{R}^2} d\sigma_\nu \iint_{p,p'} \sum (p \pm p')^\mu \left[\binom{c}{d}_p^{(\dagger)} \binom{c}{d}_{p'}^{(\dagger)} \right] \left[\binom{\bar{u}}{\bar{v}}_p \gamma^\nu \binom{u}{v}_{p'} \right] e^{\pm i(p \pm p')x}$$

$$P_e^\mu = 2\int_{\mathbb{R}^2} d\sigma_\nu \iint_{k,k'} T^{\mu\nu}(k,k') \sum_{\alpha,\beta,\dagger} \pm \left(a_\alpha^{(\dagger)}(k) \cdot a_\beta^{(\dagger)}(k') \right) e^{\pm i(k \pm k')x}$$

The red part gives rise to the covariant distribution W_ν:

$$\begin{aligned} W_\nu(\kappa) &= 2\int_{\mathbb{R}^2} d^2x\, \delta(x^2 - \tau^2)\, \theta(x^0)\, x_\nu\, e^{i\kappa x} = \\ &= \delta(\kappa^2) \quad 2\pi\, \epsilon(\kappa^0) \quad \kappa_\nu \quad + \\ &+ \theta(\kappa^2) \quad 2\pi\, \delta(\kappa^0)\, J_0(\tau\sqrt{\kappa^2}) \quad \delta_\nu^0 \quad - \\ &- \theta(\kappa^2) \quad \frac{\pi\tau}{\sqrt{\kappa^2}} \left(i\, Y_1(\tau\sqrt{\kappa^2}) + \epsilon(\kappa^0)\, J_1(\tau\sqrt{\kappa^2}) \right) \quad \kappa_\nu \quad - \\ &- \theta(-\kappa^2) \quad \frac{2i\tau}{\sqrt{-\kappa^2}}\, K_1(\tau\sqrt{-\kappa^2}) \quad \kappa_\nu \end{aligned}$$

…after contraction with $\left[\binom{\bar{u}}{\bar{v}}_p \gamma^\nu \binom{u}{v}_{p'} \right]$ resp. $T^{\mu\nu}(k,k')$,

only the second line survives!

2-Momentum operator

$$P_f^\mu = \int\!\!\int_{p,p'} \sum (p \pm p')^\mu \left[\binom{c}{d}_p^{(\dagger)} \binom{c}{d}_{p'}^{(\dagger)} \right] \left[\binom{\bar{u}}{\bar{v}}_p \gamma^\nu \binom{u}{v}_{p'} \right] W_\nu^{(*)}(p \pm p')$$

$$P_e^\mu = \int\!\!\int_{k,k'} T^{\mu\nu}(k,k') \sum\nolimits_{\alpha,\beta,\dagger} \pm \left(a_\alpha^{(\dagger)}(k) \cdot a_\beta^{(\dagger)}(k') \right) W_\nu^{(*)}(k \pm k')$$

$$W_\nu(\kappa) \to \theta(\kappa^2)\, 2\pi\, \delta(\kappa^0)\, J_0(\tau\sqrt{\kappa^2})\, \delta_\nu^0$$

Result ($\theta(0) := 1$, $J_0(0) = 1$): Same as in instant form:

Fermionic part

$$P_f^\mu = \int \frac{dp^1}{2p^0}\, p^\mu \left(c^\dagger(p^1)\, c(p^1) + d^\dagger(p^1)\, d(p^1) \right)$$

Photonic part

$$P_e^\mu = \sum_{\alpha=0}^{1} \int \frac{dk^1}{2k^0}\, k^\mu \left(a_\alpha(k)\, a_\alpha^\dagger(k') + a_\alpha^\dagger(k)\, a_\alpha(k') \right)$$

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Interactions

Interaction Lagrangian contains no derivatives:

$$\mathcal{L}_{int} = \frac{1}{2}\, e\, \overline{\psi}\, \not{A}\, \psi$$

$\Rightarrow$ Energy-momentum tensor:

$$\Theta^{\mu\nu}_{int} = -g^{\mu\nu} \mathcal{L}_{int}$$

Interactions

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$\Rightarrow$ Energy-momentum tensor:

$$\Theta^{\mu\nu}_{int} = -g^{\mu\nu} \mathcal{L}_{int}$$

$\Rightarrow$ In point form, the 2 generators of spacetime translations are interaction dependent ("dynamic"):

$$\boxed{P^{\mu}_{int}} = \int_{\sigma} d\sigma_{\nu}\, \Theta^{\mu\nu}_{int}(x) = -\int_{\sigma} d\sigma_{\nu}\, g^{\mu\nu}\, \mathcal{L}_{int}(x) =$$

$$= \boxed{-\int_{\mathbb{R}^2} 2\, d^2x\, \delta(x^2 - \tau^2)\, \theta(x^0)\, x^{\mu}\, \mathcal{L}_{int}(x)}$$

Interactions

Interaction Lagrangian contains no derivatives:

$$\mathcal{L}_{int} = \frac{1}{2}\, e\, \overline{\psi}\, \not{A}\, \psi$$

⇒ Energy-momentum tensor:

$$\Theta^{\mu\nu}_{int} = -g^{\mu\nu}\mathcal{L}_{int}$$

⇒ In point form, the generator of the Lorentz boost is interaction independent ("kinematic"):

$$J^{\mu\nu}_{int} = \begin{bmatrix} 0 & K_{int} \\ -K_{int} & 0 \end{bmatrix} = \int_\sigma d\sigma_\rho \left(x^\mu \Theta^{\rho\nu}_{int} - x^\nu \Theta^{\rho\mu}_{int}\right) =$$

$$= -\int_{\mathbb{R}^2} 2\, d^2x\, \delta(x^2 - \tau^2)\, \theta(x^0)\, \underbrace{x_\rho\left(x^\mu g^{\rho\nu} - x^\nu g^{\rho\mu}\right)}_{0} \mathcal{L}_{int}(x) \equiv 0$$

Fourier decomposition of interaction

To get the trilinear interaction term, insert decompositions of fields ψ, A^μ:

$$P^\mu_{int} = -\int_{\mathbb{R}^2} d\sigma^\mu \left(\overline{\psi}\, e\gamma^\lambda A_\lambda\, \psi \right) =$$

$$= -e \int_{\mathbb{R}^2} d^2x\, \delta\left(x^2 - \tau^2\right)\, \theta(x^0)\, x^\mu \times$$

$$\times \int_{-\infty}^{\infty} \frac{dp^1}{2\,p^0\sqrt{2\pi}} \left(c^\dagger(p)\, u^\dagger(p)\, e^{ixp} + d(p)\, v^\dagger(p)\, e^{-ixp} \right) \gamma^0 \times$$

$$\times\, \gamma^\lambda \int_{-\infty}^{\infty} \frac{dk^1}{4\pi k^0} \sum_{\kappa=0}^{1} \left(a_\kappa(k)\, \pi^\lambda_\kappa(k)\, e^{-ikx} + a^\dagger_\kappa(k)\, \pi^\lambda_\kappa(k)\, e^{ikx} \right) \times$$

$$\times \int_{-\infty}^{\infty} \frac{dp'^1}{2\,p'^0\sqrt{2\pi}} \left(c(p')\, u(p')\, e^{-ixp'} + d^\dagger(p')\, v(p')\, e^{ixp'} \right)$$

Expression with W

By use of the distribution W^μ, we obtain

$$
\begin{aligned}
P^\mu_{int} \;=\; -e \int_{-\infty}^{\infty} \frac{dp^1}{2\,p^0\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dk^1}{4\pi k^0} \int_{-\infty}^{\infty} \frac{dp'^1}{2\,p'^0\sqrt{2\pi}} \sum_{\kappa=0}^{1} \cdots \\
\Big(c^\dagger(p)\,u^\dagger(p)\,\gamma^0\,\gamma^\lambda\,a_\kappa(k)\,\pi^\lambda_\kappa(k)\;c(p')\;u(p')\,W^\mu(p-k-p')+ \\
+\,c^\dagger(p)\,u^\dagger(p)\,\gamma^0\,\gamma^\lambda\,a_\kappa(k)\,\pi^\lambda_\kappa(k)\,d^\dagger(p')\,v(p')\,W^\mu(p-k+p')+ \\
+\,d(p)\,v^\dagger(p)\,\gamma^0\,\gamma^\lambda\,a^\dagger_\kappa(k)\,\pi^\lambda_\kappa(k)\;c(p')\;u(p')\,W^\mu(p+k+p')+ \\
+\,d(p)\,v^\dagger(p)\,\gamma^0\,\gamma^\lambda\,a^\dagger_\kappa(k)\,\pi^\lambda_\kappa(k)\,d^\dagger(p')\,v(p')\,W^\mu(p+k-p')+\text{h.c.}\Big)
\end{aligned}
$$

Contractions

…and after multiplying the spinors $u^{(\dagger)}$ and $v^{(\dagger)}$ with the matrices γ^μ,

$$
\begin{aligned}
P^\mu_{int} \; = \; -e \int_{-\infty}^{\infty} \frac{dp^1}{2\,p^0\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dk^1}{4\pi k^0} \int_{-\infty}^{\infty} \frac{dp'^1}{2\,p'^0\sqrt{2\pi}} \ldots \\
\Big(\left(a_0(k)\,U(p,p') - a_1(k)\,V(p,p')\right) c^\dagger(p)\; c(p')\; W^\mu(p-k-p') + \\
+ \left(a_0(k)\,V(p,p') - a_1(k)\,U(p,p')\right) c^\dagger(p)\, d^\dagger(p')\, W^\mu(p-k+p') + \\
- \left(a_0(k)\,V(p,p') - a_1(k)\,U(p,p')\right) d(p)\; c(p')\; W^\mu(p+k+p') - \\
- \left(a_0(k)\,U(p,p') - a_1(k)\,V(p,p')\right) d(p)\; d^\dagger(p')\, W^\mu(p+k-p') \Big)
\end{aligned}
$$

(+h.c.) with "ordinary" functions

$$
\begin{aligned}
U(p,p') &:= 2p^0 p'^0 + 2p^1 p'^1 \\
V(p,p') &:= 2p^1 p'^0 + +2p^0 p'^1
\end{aligned}
$$

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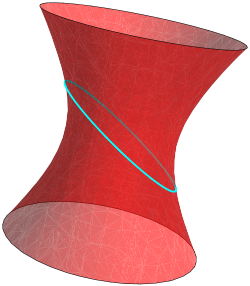
Outlook

To solve the full eigenvalue equation, $\boxed{P^{\mu}_{tot} \left|\Psi\right\rangle = p^{\mu} \left|\Psi\right\rangle}$,
non-perturbatively:

- Use de Sitter space (compact spacelike directions):
  - Covariant version of QED on a circle
  - Discrete momenta



- Algebraic manipulations (Construct operators from (multilinears of) $c^{(\dagger)}$, $d^{(\dagger)}$, $a_{\kappa}^{(\dagger)}$ to get a different Lie algebra)
- Start with one-mode problems

Outlook

Problem: Distribution W^μ not manageable
if not contracted with other vector

$\Rightarrow$ First try to solve Thirring model
(self-interacting Dirac field in 1+1 dimensions with Lagrangian

$$\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi - \frac{g}{2}\left(\overline{\psi}\gamma^\mu\psi\right)\left(\overline{\psi}\gamma_\mu\psi\right) \qquad)$$

- in position space (i.e. with operators $\psi_a^{(\dagger)}$) using hyperbolic coordinates
- using similar methods to be tested
- ...and try to apply those methods to the Schwinger model later!