Renormalization of topological parameters

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Outline

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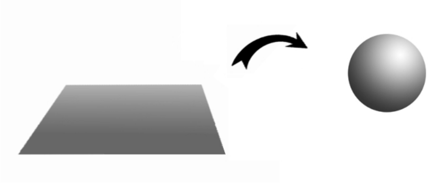
The Reluctant Topological Parameter

What are topological terms?

They represent topological properties of field configurations, like e.g. the winding number:



Not sensitive to local pertubations!

Important example in the standard model: $i\theta Q = i\theta \frac{g^2}{32\pi^2} \int d^4x \; \tilde{F}^{\alpha}_{\mu\nu} F^{\mu\nu}_{\alpha}$
Why is θ so tiny? (Strong CP-problem)

The Reluctant Topological Parameter

Is θ a running coupling?

- First expectation: Due to its topological nature it should not be renormalized by quantum fluctuations.
- However, explicit calculations revealed renormalization at extreme momentum scales[1].
- Functional Renormalization Group as non-pertubative method to determine β-function?

[1] M.A. Shifman, A.I. Vainshtein, Nucl. Phys. B 365 (1991)
A.A. Johanson, Nucl. Phys. B 376 (1992)

The Reluctant Topological Parameter

Flow of effective action is given by:

$$\dot{\Gamma}_k = \tfrac{1}{2}\mathrm{Tr}\left\{\frac{\dot{\mathcal{R}}_k}{\mathcal{R}_k + \Gamma_k^{(2)}}\right\}$$

The effective action contains a topological term $i\theta_k Q$.
But topological quantum numbers are invariant under local field variations, i.e.

$$Q^{(2)} = 0$$

$\Longrightarrow$ There is apparently no renormalization of θ_k.

The Reluctant Topological Parameter

Idea by Martin Reuter [2]:

What happens when we regard the topological term as a limit of a more general problem?

$$i\theta \frac{g^2}{32\pi^2} \int d^4x \; \phi(x) \; \tilde{F}^{\alpha}_{\mu\nu} F^{\mu\nu}_{\alpha}$$

"Axion" field $\phi(x)$, which we assume to be constant at the end: $\phi(x) \to 1$.

[2] Mod. Phys. Lett. A 12 (1997)

The Reluctant Topological Parameter

This modification leads to a renormalization of θ determined by two jumps:

for k : $0 < k < \infty$

1. in the UV: $$\theta(k) = \left[1 - \frac{T(G)}{4\pi^2} g^2(\infty) \left(1 + \tfrac{1}{4}(1-\alpha)\right) \left\{1 - \tfrac{1}{2}\xi\eta(\infty)\right\}\right] \theta(\infty)$$

2. in the IR: $$\theta(0) = \left[1 - \frac{T(G)}{4\pi^2} g^2(0) Z_F(0)^{-1}\right] \theta(k)$$

The Reluctant Topological Parameter

Is the introduction of an auxiliary field the right way to deal with topological parameters?

We should obtain a better understanding of this approach by applying it to a simpler model.

O(3) nonlinear sigma model in two dimensions:

$$S = \tfrac{1}{2}\zeta \int d^2x \; h_{ab}(\varphi)\, \partial^\mu \varphi^a \partial_\mu \varphi^b \; + i\tfrac{1}{2}\theta \int d^2x \; \sqrt{h}\epsilon_{ab}\epsilon^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi^b$$

$$h_{ab}(\varphi) = \frac{1}{(1+\varphi^2)^2} \qquad\qquad (a, b, \mu, \nu = 1, 2)$$

Moreover, the model is interesting in its own right. E.g. there is a mapping between spin-S chains and σ models with $\theta = 2\pi S$.

Outline

Renormalization of the $O(3)$ Nonlinear σ Model

The calculation of $\Gamma^{(2)}$ is most conveniently done with a **background field expansion**, i.e. a covariant expansion of φ in quantum fluctuations ξ around a background $\bar{\varphi}$:

$$\Gamma_k = ... + \tfrac{1}{2}\zeta_k \int d^2x\; h_{ab}\nabla_\mu\xi^a\nabla^\mu\xi^b + R_{abcd}\partial_\mu\bar{\varphi}^a\partial^\mu\bar{\varphi}^d\xi^b\xi^c$$
$$+ \tfrac{i}{2}\theta_k \int d^2x\; \sqrt{h}\epsilon_{ab}\epsilon^{\mu\nu}\phi\left(\nabla_\mu\xi^\alpha\nabla_\nu\xi^\beta + R^a_{cde}\partial_\mu\bar{\varphi}^e\partial_\nu\bar{\varphi}^b\xi^c\xi^d\right) + \mathrm{O}(\xi^3)$$

where $\nabla_\mu\xi^a = \partial_\mu\xi^a + \Gamma^a_{bc}\partial_\mu\bar{\varphi}^b\xi^c$.

We will evaluate the flow equation at $\xi = 0$ and hence require

$$\Gamma^{(2)}_k\Big|_{\xi=0} = \underbrace{-\zeta_k(\nabla_\mu\nabla^\mu)_{ab} + \zeta_k R_{acdb}\partial_\mu\bar{\varphi}^c\partial^\mu\bar{\varphi}^d}_{:=-\zeta_k\tilde{\Delta}} - \underbrace{i\theta_k\sqrt{h}\epsilon_{ab}\epsilon^{\mu\nu}(\partial_\mu\phi)\nabla_\nu}_{B(\phi)}$$

Renormalization of the $O(3)$ Nonlinear σ Model

We choose a regulator of the form $\mathcal{R}_k = \zeta_k R_k(-\tilde{\Delta})$

$$\dot{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\left\{\frac{\zeta_k(\dot{R}_k - \eta_k R_k)}{\zeta_k R_k - \zeta_k\tilde{\Delta} - B(\phi)}\right\}, \text{ where } \eta_k = -\frac{\dot{\zeta}_k}{\zeta_k}$$
$$= \frac{1}{2}\mathrm{Tr}\left\{\frac{(\dot{R}_k - \eta_k R_k)}{R_k - \tilde{\Delta}} + \zeta_k^{-1}B(\phi)f(-\tilde{\Delta}) + O(\phi^2)\right\}$$

It will be useful to employ an inverse Laplace transform and a heat kernel expansion: $f(-\tilde{\Delta}) = \int_0^\infty ds\ \tilde{f}(s)\exp\left(-s(-\tilde{\Delta})\right)$.

First part was already solved[3] and yields a renormalization of ζ_k:

$$\dot{\zeta}_k = \frac{1}{2\pi}\left(1 - \frac{1}{4\pi\zeta_k}\right)^{-1}$$

The flow of the inverse coupling $\frac{1}{g^2} = \zeta$ reads: $\quad \dot{g} = -\frac{1}{4\pi}g^3\left(1 - \frac{g^2}{4\pi}\right)^{-1}$

[3]A. Codello, R. Percacci, Phys. Lett. B 672 (2009)

Renormalization of the $O(3)$ Nonlinear σ Model

In order to deal with $\zeta_k^{-1} B(\phi) f(-\tilde{\Delta})$ we need the off-diagonal elements of the heat kernel expansion:

$$
\begin{aligned}
&\mathrm{Tr}\{\zeta_k^{-1} B(\phi) f(-\tilde{\Delta})\} \\
&= \zeta_k^{-1} \int_0^\infty ds \tilde{f}(s) \int d^2x d^2y \; i\theta_k \sqrt{h} \epsilon_{ab} \epsilon^{\mu\nu} \underbrace{<x|(\nabla_\mu \phi) \nabla_\nu |y>^{bc}}_{=\delta(x-y)\nabla_\mu \phi(y) \nabla_\nu(y)} \underbrace{< y|e^{-s(-\tilde{\Delta})}|x >^a_c}_{:=\Omega(y,x,s)} \\
&= -i \frac{\theta_k}{\zeta_k} \int_0^\infty ds \tilde{f}(s) \int d^2x d^2y \sqrt{h} \epsilon_{ab} \epsilon^{\mu\nu} \phi(y) \delta(x-y) \\
&\qquad \times \left[\tfrac{1}{2} H_{\mu\nu}(y) \Omega(y,x,s) + \nabla_\mu(x) \nabla_\nu(y) \Omega(y,x,s) \right]^{ba},
\end{aligned}
$$

where $(H_{\mu\nu})_{ab} = [\nabla_\mu, \nabla_\nu]_{ab} = R_{abcd} \partial_\mu \bar{\varphi}^c \partial_\nu \bar{\varphi}^d$.

Renormalization of the $O(3)$ Nonlinear σ Model

Ansatz for **heat kernel expansion** in 2-dim. flat spacetime:

$$\Omega(x,y,s) = <x|e^{-s(-\tilde{\Delta})}|y> = \frac{1}{4\pi s} e^{-\frac{|x-y|^2}{4s}} \sum_{k=0}^{\infty} s^k c_k(x,y).$$

It has to fulfil: $(\frac{d}{ds} - \tilde{\Delta})\Omega(x,y,s) = 0$

$$\Longrightarrow \quad k c_k + (x^\mu - y^\mu)\nabla_\mu c_k - \tilde{\Delta} c_{k-1} = 0$$

Renormalization of the $O(3)$ Nonlinear σ Model

Solution for the heat kernel coefficients:

$$c_0(x,y) = e^{-\int_y^x dz^\mu \; \Gamma \partial_\mu \bar{\varphi}}$$

$$c_k(x,y) = e^{-\int_y^x dz \; \Gamma \partial \bar{\varphi}} \int_0^1 d\lambda \; \lambda^{k-1} \left(e^{\int_y^x dz \; \Gamma \partial \bar{\varphi}} \right)^{*\lambda} \left(\tilde{\Delta} c_{k-1} \right)^{*\lambda},$$

where $[A(x,y)]^{*\lambda}$ indicates that $A(x,y)$ is expanded around y in powers of $(x-y)^\mu$ and all factors $(x-y)^\mu$ in this expansion are multiplied by λ.

Renormalization of the $O(3)$ Nonlinear σ Model

With this expansion, we get

$$\begin{aligned}&\tfrac{1}{2}H_{\mu\nu}(y)\Omega(y,x,s)+\nabla_\mu(x)\nabla_\nu(y)\Omega(y,x,s)\\&=\frac{1}{8\pi s^2}e^{-\frac{|x-y|^2}{4s}}H_{\mu\lambda}(x)(x-y)^\lambda(x-y)_\nu\ c_0(x,y)+\mathrm{O}(\partial\varphi^3)\end{aligned}$$

Comparison of antisymmetric quadratic structures in flow equation:

If we plug this into the flow equation and use some features of the two-dimensional geometries, we get:

$$\begin{aligned}&\frac{i}{2}\partial_t\theta_k\int d^2x\ \sqrt{h}\epsilon^{\mu\nu}\epsilon_{ab}\ \phi(x)\partial_\mu\bar{\varphi}^a(x)\partial_\nu\bar{\varphi}^b(x)\\&=\frac{i}{2}\ \frac{\theta_k}{\zeta_k}\int d^2x\ \phi(x)\lim_{x\to y}\int_0^\infty ds\ \tilde{f}(s)\frac{(x-y)^2}{8\pi s^2}e^{-\frac{|x-y|^2}{4s}}\sqrt{h}\epsilon_{ab}\epsilon^{\mu\nu}\partial_\mu\bar{\varphi}^a\partial_\nu\bar{\varphi}^b\end{aligned}$$

We have to be cautious with taking the limit!

Renormalization of the $O(3)$ Nonlinear σ Model

What is the structure of the inverse Laplace transform $\tilde{f}$?

$$\tilde{f}(s) = \mathcal{L}^{-1}\left[\frac{\dot{R}_k(r) - \eta_k R_k(r)}{(r + R_k(r))^2}\right](s) \equiv \sigma(\mathbf{k^2 s})$$

$$\sigma(k^2 s) = \mathcal{L}^{-1}\left[\partial_t (r + R_k(r))^{-1}\right](s) - \eta_k \mathcal{L}^{-1}\left[\frac{R_k(r)}{(r + R_k(r))^2}\right](s)$$

$$= \partial_t \sigma_1(k^2 s) - \eta_k \sigma_2(k^2 s)$$

Renormalization of the $O(3)$ Nonlinear σ Model

Running of θ from UV down to a finite scale

$$\theta(\infty) - \theta(k_0^2)$$

$$= \lim_{z\to 0} \int_{k_0^2}^{\infty} dk^2 \int_0^{\infty} ds \frac{z^2}{8\pi s^2} e^{-\frac{z^2}{4s}} \left[\partial_{k^2}\sigma_1(k^2 s) - \tfrac{1}{2}\eta_k \frac{1}{k^2}\sigma_2(k^2 s) \right] \frac{\theta}{\zeta}(k^2)$$

$$= \lim_{z\to 0} \int_0^{\infty} ds \frac{1}{2\pi s^2} e^{-\frac{1}{s}} \int_{k_0^2}^{\infty} dk^2 \left[\partial_{k^2}\sigma_1(\tfrac{1}{4}z^2 k^2 s) - \tfrac{1}{2}\eta_k \frac{1}{k^2}\sigma_2(\tfrac{1}{4}z^2 k^2 s) \right] \frac{\theta}{\zeta}(k^2)$$

$$= \lim_{z\to 0} \int_0^{\infty} ds \frac{1}{2\pi s^2} e^{-\frac{1}{s}} \int_{\frac{1}{4}k_0^2 z^2 s}^{\infty} dp^2 \left[\partial_{p^2}\sigma_1(p^2)\, \frac{\theta}{\zeta}\left(\frac{4p^2}{z^2 s}\right) - \tfrac{1}{2}\frac{1}{p^2}\sigma_2(p^2)\, \eta\frac{\theta}{\zeta}\left(\frac{4p^2}{z^2 s}\right) \right]$$

$$= \underbrace{\int_0^{\infty} ds \frac{1}{2\pi s^2} e^{-\frac{1}{s}}}_{=\sqrt{\pi}/4} \int_0^{\infty} dp^2 \left[\partial_{p^2}\sigma_1(p^2)\frac{\theta}{\zeta}(\infty) - \tfrac{1}{2}\frac{1}{p^2}\sigma_2(p^2)\eta\frac{\theta}{\zeta}(\infty) \right]$$

Renormalization of the $O(3)$ Nonlinear σ Model

$$\theta(\infty) - \theta(k_0^2) = \sqrt{\pi}/4 \int_0^\infty dp^2 \, \left[\partial_{p^2} \sigma_1(p^2) \frac{\theta}{\zeta}(\infty) - \tfrac{1}{2} \frac{1}{p^2} \sigma_2(p^2) \eta \frac{\theta}{\zeta}(\infty) \right]$$

It looks like a jump in the UV, but $\zeta_k \to \infty$ for $k \to \infty$.
$\Rightarrow$ There is no running of θ:

$$\boldsymbol{\theta(\mathbf{k_0^2}) = \theta(\infty)}$$

(for any $k_0 > 0$ and $\theta(\infty) < \infty$)

But what is the behavior at $k_0 = 0$?

Renormalization of the $O(3)$ Nonlinear σ Model

Treatment of the extreme infrared:

- reformulate r.h.s. of flow equation in a “fermionic” language
- utilize cancellation of modes of opposite chirality
- application of index theorem

Renormalization of the $O(3)$ Nonlinear σ Model

Translation into a "fermionic language"

Gamma matrices: $\Gamma_\mu = \begin{bmatrix} 0 & \Omega_\mu \\ \Omega_\mu^T & 0 \end{bmatrix}$

with $\Omega_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^a{}_b$ and $\Omega_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta^a{}_b$.

"Chirality operator": $\Gamma_* = -\begin{bmatrix} \epsilon^a{}_b & 0 \\ 0 & \epsilon^a{}_b \end{bmatrix} \Gamma_1 \Gamma_2 = \begin{bmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{bmatrix}$

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}\mathbf{1}_4, \quad \{\Gamma_*, \Gamma_\mu\} = 0, \quad \Gamma_*^2 = \mathbf{1}_4$$

Dirac operator $\not{D} = \Gamma_\mu \nabla^\mu$ and $D := \Omega_\mu \nabla^\mu$.

$(\Omega_\mu)^a{}_b$ and the connection $\Gamma^a{}_{cb}\partial_\mu\varphi^c$ commute.

Renormalization of the $O(3)$ Nonlinear σ Model

The new matrices provide useful relations like e.g.

$$\epsilon^{a}{}_{b}\epsilon^{\mu\nu}\nabla_\mu\nabla_\nu = \frac{1}{2}\left(\Omega_\mu\Omega_\nu^T - \Omega_\mu^T\Omega_\nu\right)\nabla^\mu\nabla^\nu = \frac{1}{2}(DD^T - D^T D)$$

It is furthermore convenient to evaluate the flow equation in an instanton background:

$$\partial^\mu\bar{\varphi}^a = \epsilon^{\mu\rho}\epsilon^{a}{}_{b}\partial_\rho\bar{\varphi}^b$$

It can be shown that in such a background

$$\tilde{\Delta} = D^T D$$

Renormalization of the $O(3)$ Nonlinear σ Model

Having all this, one can rewrite the relevant part of the flow equation as

$$\mathrm{Tr}_2\{\zeta_k^{-1}B(\phi)f(-\tilde{\Delta})\} = \tfrac{i}{2}\theta_k\zeta_k^{-1}\int d^2x\sqrt{h}\epsilon^a{}_b\epsilon^{\mu\nu} < x|(\nabla_\mu\phi)\nabla_\nu f(-D^TD)|x >^b{}_a$$

$$= -\tfrac{i}{4}\theta_k\zeta_k^{-1}\mathrm{Tr}_2\left\{\phi\left[DD^Tf(-DD^T) - D^TDf(-D^TD)\right]\right\} + \mathrm{O}(\partial\varphi^3)$$

$$= -\tfrac{i}{4}\theta_k\zeta_k^{-1}\mathrm{Tr}_4\left\{\phi\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}DD^T & 0\\ 0 & D^TD\end{bmatrix}\; f\left(\begin{bmatrix}-DD^T & 0\\ 0 & -D^TD\end{bmatrix}\right)\right\} + \mathrm{O}(\partial\varphi^3)$$

$$= -\tfrac{i}{4}\theta_k\zeta_k^{-1}\mathrm{Tr}_4\left\{\phi\Gamma_*\not{D}^2 f(-\not{D}^2)\right\} + \mathrm{O}(\partial\varphi^3).$$

Modes of positive and negative chirality cancel each other.
$\Rightarrow$ Only zero-modes are relevant!

Renormalization of the $O(3)$ Nonlinear σ Model

Taking this cancellation into account and integrating the flow from $k = 0$ up to an arbitrarily small k_0 while looking at the zero-modes of $\not{D}^2$, we get:

$$(\theta_{k_0^2} - \theta_0) \int d^2x\sqrt{h}\, \phi\, \epsilon_{ab}\epsilon^{\mu\nu}\partial_\mu\bar{\varphi}^a\partial_\nu\bar{\varphi}^b = \tfrac{1}{2}\zeta_0^{-1}\theta_0 \left(1 - \tfrac{1}{35}\eta_0\right) \mathrm{Tr}_4 \left\{\phi\Gamma_*\right\}$$

$\mathrm{Tr}_4 \{\phi\Gamma_*\}$ has to be regulated in order to be well-defined: $\lim_{s\to 0} \mathrm{Tr}_4 \left\{\phi\Gamma_* e^{s\not{D}^2}\right\}$
This index can be calculated by a heat kernel expansion.

$$\begin{aligned}&(\theta_{k_0^2} - \theta_0) \int d^2x\sqrt{h}\, \phi\, \epsilon_{ab}\epsilon^{\mu\nu}\partial_\mu\bar{\varphi}^a\partial_\nu\bar{\varphi}^b \\ &= -\tfrac{1}{4\pi}\zeta_0^{-1}\theta_0 \left(1 - \tfrac{1}{35}\eta_0\right) \int d^2x\sqrt{h}\, \phi\, \epsilon_{ab}\epsilon^{\mu\nu}\partial_\mu\bar{\varphi}^a\partial_\nu\bar{\varphi}^b\end{aligned}$$

$$\Longrightarrow \theta_0 = \frac{1}{1 - (1 - \frac{1}{35}\eta_0)\frac{1}{4\pi}\zeta_0^{-1}}\ \theta_\infty$$

Outline

Conclusion and Discussion

$$\theta_0 = \frac{1}{1 - (1 - \frac{1}{35}\eta_0)\frac{1}{4\pi}\zeta_0^{-1}}\ \theta_\infty$$

- The topological parameter θ is constant from the UV down to any finite scale, but jumps in the extreme IR.
- The idea to reveal the topological flow by introducing an auxiliary field $\phi(x)$ works not only for Yang-Mills theory but also for other models.
- However, the results should make us sceptical...

Conclusion and Discussion

$$\theta_0 = \frac{1}{1 - \left(1 - \frac{1}{35}\eta_0\right)\frac{1}{4\pi}\zeta_0^{-1}}\ \theta_\infty$$

Shortcomings of the approach:

- result is not periodic in 2π
- $\dot{\zeta}$ does not depend on θ
- we know from other analytic studies that mass spectrum depends on θ