Binet-Legendge ellipsoid in conformal Finsler geometry

Vladimir S. Matveev (Jena)

Based on the paper arXiv:1104.1647 joint with Marc Troyanov
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Abstract: I show a simple construction from convex geometry that solves many named problems in Finsler geometry
Definition of Finsler metrics:
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(a) $F(\lambda \cdot v) = \lambda \cdot F(v)$,  
(b) $F(u + v) \leq F(u) + F(v)$,  
(c) $F(v) = 0 \iff v = 0$. 
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$E : R^n \to R$ of the form
$E(v) = \sqrt{\sum_{i,j} a_{ij} v^i v^j}$,
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$F(x, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is a norm, i.e., satisfies

(a), (b), (c).
How to visualize Finsler metrics

Repeat: Minkowski norm is a function $B : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with

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It is known (Minkowski) that the unit ball determines the norm uniquely:

for a given convex body $K \in \mathbb{R}^n$ such that $0 \in \text{int}(K)$ there exists an unique norm $B$ such that $K = \{ x \in \mathbb{R}^n \mid B(x) \leq 1 \}$. 

There exists a unique norm such that (the convex body) $K$ is the unite ball in this norm
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Thus, in order to make a picture of a Finsler metric it is sufficient to draw unit balls at the tangent spaces.

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**Minkowski metric on** $\mathbb{R}^n$: $F(x, v) = B(v)$ for a certain norm $B$, i.e., the metric is invariant w.r.t. the standard translations of $\mathbb{R}^n$. 
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![Riemannian 2D metric: all unite balls are ellipses](image)

**Minkowski metric on** \( \mathbb{R}^n \): \( F(x, v) = B(v) \) for a certain norm \( B \), i.e., the metric is invariant w.r.t. the standard translations of \( \mathbb{R}^n \).

![Minkowski 2D metric](image)

**Arbitrary Finsler metric on** \( \mathbb{R}^n \): \( F(x, v) = B(v) \) for a certain norm \( B \), i.e., the metric is invariant w.r.t. the standard translations of \( \mathbb{R}^n \).

![Finsler 2D metric: unite balls are convex; that's all](image)
I show a simple trick in Finlser geometry
Plan — Main messages of my talk

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- I show a simple trick in Finlser geometry
- I demonstrate that the trick is extremely effective in the Finlser geometry: I show a bunch of named problems that were solved.
- Finlser geometers always emphasize that Finsler metric can be used in the description of nature. Could the trick be applied there?
Main Trick

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  metric $g$, then $g_F = g$
- If two Finsler metrics $F_1$ and $F_2$ are conformally equivalent, i.e., if
  $F_1(x, \xi) = \lambda(x)F_2(x, \xi)$ for some function $\lambda : M \to R$, then the
  corresponding Riemannian metrics are also conformally equivalent with
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This allows to use the results and methods from (much better developed) Riemannian geometry to Finsler geometry. I will explain how and show many application
Construction of the (Binet-Legendre) Euclidean structure in every tangent space
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We take an arbitrary linear volume form $\Omega$ in $V$, i.e., $\Omega = \text{const} \cdot dx^1 \wedge \ldots \wedge dx^n$, and construct contravariant bilinear form $g^* : V^* \times V^* \to R$ (where $V^*$ is the dual vector space to $V$), i.e., $g^{1j}$ by

$$g^*(\xi, \nu) := \frac{1}{\text{Vol}_\Omega(K)} \int_K \xi(x) \nu(x) d\Omega$$
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Example. Let us calculate $g^{12}$ by this formula: in this case $\xi(k) = \xi(x_1, \ldots, x_n) = x_1$, $\nu(k) = \nu(x_1, \ldots, x_n) = x_2$, and $g^{12} = \frac{1}{\int_K 1d\Omega} \int_K x_1x_2 d\Omega$. 
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$g^*(\xi, \nu) := \frac{1}{\text{Vol}_\Omega(K)} \int_K \xi(k) \nu(k) d\Omega$

Evidently, $g$ is a well-defined Euclidean structure.
\[ g^*(\xi, \nu) := \frac{1}{Vol_\Omega(K)} \int_K \xi(k) \nu(k) d\Omega \]

**Evidently,** \( g \) is a well-defined Euclidean structure

- it does not depend on \( \Omega \) (because the only freedom is choosing \( \Omega \), multiplication by a constant, does not influence the result),
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- \( g' \) constructed by \( K' := \frac{1}{\lambda} \cdot K \) is given by \( g' = \lambda^2 \cdot g \)
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Thus, by a Finsler metric $F$, we canonically constructed a Euclidean structure on every tangent space, i.e., a Riemannian metric $g_F$. If the Finsler metric is smooth, then the Riemannian metric is also smooth.
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This metric has the following property: $g_{\lambda \cdot F} = \lambda^2 \cdot g_F$.

In particular, if $\phi$ is isometry, similarity, or conformal transformation of $F$, it is an isometry, similarity, or conformal transformation of $g_F$. 
General schema how to apply the trick

1. One use it to reformulate the problem in Finsler geometry as the problem in better studied Riemannian geometry
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3. And uses the additional information to solve the initial problem.
First application: Wang’s Theorem for all dimensions.
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**Theorem.** Let \((M^n, F)\) be a \(C^2\)-smooth connected Finsler manifold. If the dimension of the space of Killing vector fields of \((M, F)\) is greater than \(\frac{n(n-1)}{2} + 1\), then \(F\) is actually a Riemannian metric.
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Proof. Let \(F\) be a Finsler metrics admitting \(\frac{n(n-1)}{2} + 2\) Killing vector fields. Consider the Riemannian metric \(g_F\). The Killing vector fields of \(F\) are Killing vector fields of \(g_F\).
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Now, metrics admitting “many” Killing vector fields are well-studied: It is known (and is a nontrivial mathematical statement due to (Yano, Kato)) that for metrics with at least \(\frac{n(n-1)}{2} + 2\) Killing vector field the isotropy subgroup of the isometry group acts transitively on the unite sphere in the tangent space.

The isotropy group corresponding to \(x \in M\) consists of all isometries taking \(x\) to \(x\). It preserves the Finsler and the Riemannian metric. Then, it preserves the quotient \(F(\xi)^2/g(\xi, \xi)\). Since it acts transitively, \(F(\xi) = \text{const} \sqrt{g(\xi, \xi)}\), i.e., \(F\) is a Riemannian metric.
The Liouville Theorem for Minkowski spaces and the solution to a problem by Matsumoto.

**Theorem.** Let \((V, F)\) be an non-euclidean Minkowski space. If \(\phi : U_1 \rightarrow U_2\) is a conformal map between two domains \(U_1 \subset V\) and \(U_2 \subset V\), then \(\phi\) is (the restriction of) a similarity, that is the composition of an isometry and a homothety \(x \mapsto \text{const} \cdot x\).
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**Remark.** Theorem generalizes classical result of Liouville for Minkowski metrics: Liouville has shown 1850 that every conformal transformation of the standard \((\mathbb{R}^n_{\geq 3}, g_{\text{euclidean}})\) is a similarity or a Möbius transformation, i.e., a composition of a similarity and an inversion. We see that for noneuclidean Finsler metrics only similarities are allowed.
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Theorem answers the question of Matsumoto 2001 and will be uses below.
Proof of: Every conformal mapping of a Minkowski space is a similarity

Proof for $\dim(M) > 2$. I will use: if $\phi$ is a conformal transformation of $F$, then it is a conformal transformation of $g_F$. Moreover, if $\phi$ is a conformal transformation of $F$ and similarity of $g_F$, then it is a similarity of $F$. 


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We consider the metric $g_F$. 
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Compact manifolds with a similarity structure have been topologically classified by N. H. Kuiper (1950) and D. Fried (1980): they are either Bieberbach manifolds (i.e. $R^n/\Gamma$, where $\Gamma$ is some crystallographic group of $R^n$), or they are Hopf-manifolds i.e. compact quotients of $R^n \setminus \{0\} = S^{n-1} \times R_+$ by a group $G$ which is a semi-direct product of an infinite cyclic group with a finite subgroup of $O(n+1)$. 
**Def.** A $C^1$-map $f : (M, F) \to (M', F')$ is a *similarity* if there exists a constant $a > 0$, $a \neq 1$ (called the *dilation constant*) such that $F(f(x), df_x(\xi)) = a \cdot F(x, \xi)$ for all $(x, \xi) \in TM$. 
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Theorem. Let $(M, F)$ be a forward complete connected $C^0$-Finsler manifold (the manifold $M$ is of class $C^1$, the metric $F$ is $C^0$). If there exists a non isometric self-similarity $f : M \rightarrow M$ of class $C^1$, then $(M, F)$ is a Minkowski space.
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**Remark.** In the case of smooth Finsler manifolds, Theorem is known. A first proof was given by Heil and Laugwitz in 1974, however R. L. Lovas, and J. Szilasi found a gap in the argument and gave a new proof in 2009.
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Because the group $O(n)$ is compact. Hence, any sequence of the form $\psi, \psi^2, \psi^3, \ldots$, has a subsequence converging to $Id$. Thus, in the arguments on the previous slide we can take the subsequence $k \to \infty$ such that

$$
(\psi \circ \phi)^k \phi \circ \psi \equiv \psi \circ \phi \sim Id
$$

is “almost” $\phi^k$, and the proof works.
Examples of conformal transformations and its full description
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(iii) Let $g$ be the standard (Riemannian) metric on the standard sphere $S^n$. Then, the standard Möbius transformations of $S^n$ are conformal transformations of every metric $F := \lambda \cdot g$. 
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Theorem (Finsler version of conformal Lichnerowicz conjecture).
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Examples of conformal transformations and its full description

(i) If $\phi : M \to M$ is an isometry for $F$, and $\lambda : M \to \mathbb{R}_{>0}$ is a function, then $\phi$ is a conformal transformation of $F_1 := \lambda \cdot F$.

(ii) Let $F_m$ be a Minkowski metric on $\mathbb{R}^n$. Then, the mapping $x \mapsto \text{const} \cdot x$ (for $\text{const} \neq 0$) is a conformal transformation. Moreover, it is also a conformal transformation of $F := \lambda \cdot F_m$. Moreover, if $\psi$ is an isometry of $F_m$, then $\psi \circ \phi$ is a conformal transformation of every $F := \lambda \cdot F_m$.

(iii) Let $g$ be the standard (Riemannian) metric on the standard sphere $S^n$. Then, the standard Möbius transformations of $S^n$ are conformal transformations of every metric $F := \lambda \cdot g$.

**Theorem (Finsler version of conformal Lichnerowicz conjecture).** That’s all: Let $\phi$ be a conformal transformation of a connected (smooth) Finsler manifold $(M^{n \geq 2}, F)$. Then $(M, F)$ and $\phi$ are as in Examples (i, ii, iii) above.
Even in the Riemannian case, Theorem above is nontrivial
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Riemannian theorem (proved before by Alekseevsky 1971, Schoen 1995, (Lelong)-Ferrand 1996) Let \( \phi \) be a conformal transformation of a connected RIEMANNIAN manifold \((M^{n \geq 2}, g)\). Then for a certain \( \lambda : M \to R \) one of the following conditions holds

(a) \( \phi \) is an isometry of \( \lambda \cdot g \), or
(b) \((M, \lambda \cdot g)\) is \((R^n, g_{\text{flat}})\),
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Remark. In the pseudo-Riemannian case, the analog of Theorem is wrong (a counterexample in signature \((2, n - 1)\) of Frances). In the Lorenz signature, the question is still open.
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**Trivial case**: $\phi$ is an isometry of a certain $\lambda \cdot g_F$. Then, it is an isometry of $\lambda^2 \cdot F$.

**Case $R^n$**: After the multiplication of $F$ by an appropriate function, $g_F$ is the standard Euclidean metric, and $\phi$ is a similarity of $g_F$. 
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(Trivial case): $\phi$ is an isometry of a certain $\lambda \cdot g_F$. Then, it is an isometry of $\lambda^2 \cdot F$.

(Case $R^n$): After the multiplication of $F$ by an appropriate function, $g_F$ is the standard Euclidean metric, and $\phi$ is a similarity of $g_F$. Then, as we have shown above, $F$ is Minkowski.

(Case $S^n$): After the multiplication $F$ by an appropriate function, $g_F$ is the standard metric on the sphere, and $\phi$ is a möbius transformation of the sphere.
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One can generalize our proof for $R^n$ to the case $S^n$ (the principal observation that sequence of the points $p, \phi(p), \phi^2(p), ...$ converges to a fixed point is also true on the sphere; the analysis is slightly more complicated).
Solution of Deng-Hou conjecture
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**Def.** The Finsler manifold \((M, F)\) is called *locally symmetric*, if for every point \(x \in M\) there exists \(r = r(x) > 0\) (called the symmetry radius) and an isometry \(\tilde{I}_x : B_r(x) \to B_r(x)\) (called the reflexion at \(x\)) such that \(\tilde{I}_x(x) = x\) and \(d_x(\tilde{I}_x) = -\text{id} : T_xM \to T_xM\).
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**Theorem.** Let $(M, F)$ be a $C^2$-smooth Finsler manifold. If $(M, F)$ is locally symmetric, then $F$ is Berwald.
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**Remark.** This theorem answers positively a conjecture of Deng-Hou 2009, where it has been proved for globally symmetric spaces.
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**Remark.** Locally symmetric Berwald metrics are easy to construct — take the Levi-Civita connection \(\nabla\) of a locally symmetric Riemannian manifolds, choose a reversible norm at one \(T_xM\) invariant with respect to the holonomy group, and extend the norm to all points \(y \in M\) with the help of parallel transport. The obtained Finsler metric is then automatically invariant w.r.t. the reflections.
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Corollary. Every locally symmetric \(C^2\)-smooth Finsler manifold is locally isometric to a globally symmetric Finsler space.
Proof under the additional assumption that the symmetry radius is locally bounded from zero.
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The Binet-Legendge metric is a locally symmetric metric. Let us now show that the metrics $g_F$ and $F$ are affinely equivalent, that is, for every arclength parameterised $F$-geodesic $\tilde{\gamma}$ there exists a nonzero constant $c$ such that $\tilde{\gamma}(c \cdot t)$ is an arclength parameterised $g_F$-geodesic.
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Proof under the additional assumption that the symmetry radius is locally bounded from zero.

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It is sufficient to show that for every sufficiently close points $x, y \in M$ the midpoints of the geodesic segments $\gamma$ and $\tilde{\gamma}$ in the metrics $g_F$ and $F$ connecting the points $x$ and $y$ coincide.

Indeed, if it is true, then the geodesics $\gamma$ and $\tilde{\gamma}$ coincide on its dense subset implying they coincide.
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Conformal invariants of Finsler metrics

**Def.** Conformal invariants of $(M, F)$ are functions on $M$ canonically constructed by $F$ and invariant w.r.t. conformal change $F \rightarrow \lambda(x) \cdot F$. In the Riemannian case, it is hard to construct them. In the Finsler case, the metric $g_F$ helps:
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We define conformal invariants via the Steiner Formula:

\[
\text{Vol}(B_F + t \cdot B) = \sum_{j=0}^{n} \binom{n}{j} W_j(B_F) t^j,
\]

where \(B_F\) is the 1-ball in \(F\), \(B\) is the 1-Ball in \(g_F\), \(\text{Vol}\) is in \(g_F\), and everything is done in one tangent space.

These numbers \(W_j(x)\) depend only on \(F|_{T_x M}\) and are the same for \(F\) and \(\lambda(x) \cdot F\)!!!
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These numbers \(W_j(x)\) depend only on \(F_{|T_x M}\) and are the same for \(F\) and \(\lambda(x) \cdot F\)!!!!

One can construct two more invariants:

\[
M(x) = \max_{\xi \in T_x M} \frac{F(x, \xi)}{\sqrt{g(\xi, \xi)}} \quad \text{and} \quad m(x) = \min_{\xi \in T_x M} \frac{F(x, \xi)}{\sqrt{g(\xi, \xi)}}.
\]

Thus, in the generic case we obtain \(n + 2\) “easy to calculate” scalar invariants.
What to do next: possible applications in sciences
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We suggest to replace the Finsler metric $F$ by a Riemannian metric $g_F$, and then to analyze it. Of course, we lose a lot of information, but get an object which is easier to investigate.
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Note that we even do not require that the “unit ball” is smooth and convex.
When this approach should be used? And how?

- One should have something that could be a Finlser metric – for example a field of convex bodies.
- In the best case one should not have a background Riemannian or Euclidean metric.
- The construction $F \to g_F$ gives us an invariant objects whose properties are much simple than of the initial object and which can be studied by the Riemannian mashinary.
Thank you for your attention!!!