The boundary value problem for a rigidly rotating disc of charged dust

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Research Training Group "Quantum and Gravitational Fields"

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Table of Contents

1 Motivation
Table of Contents

1 Motivation

2 Basic concepts
   - The Einstein-Maxwell equations
   - The Model of Matter
   - Metric and Four-potential
   - Wave and Field equations
   - The Co-rotating frame
The boundary value problem for a rigidly rotating disc of charged dust

Table of Contents

1 Motivation

2 Basic concepts
   • The Einstein-Maxwell equations
   • The Model of Matter
   • Metric and Four-potential
   • Wave and Field equations
   • The Co-rotating frame

3 The boundary conditions
   • The disc case
   • Calculation
   • Physical interpretation
Solution to the rigidly rotating disc of dust
[Neugebauer and Meinel, 1993]
Solution to the rigidly rotating disc of dust
[Neugebauer and Meinel, 1993]
- continuous black hole limit (extreme Kerr metric)
Solution to the rigidly rotating disc of dust
[Neugebauer and Meinel, 1993]
- continuous black hole limit (extreme Kerr metric)

Electrically counterpoised dust (ECD) configurations
(Papapetrou-Majundar class)
- static solutions (including charged discs)
- continuous black hole limit (extreme Reissner-Nordström metric)
[Meinel and Hütten, 2011]
Solution to the rigidly rotating disc of dust
[Neugebauer and Meinel, 1993]
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Electrically counterpoised dust (ECD) configurations (Papapetrou-Majundar class)
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  [Meinel and Hütten, 2011]

Getting an new exact solution with a good physical interpretation
Solution to the rigidly rotating disc of dust [Neugebauer and Meinel, 1993]
- continuous black hole limit (extreme Kerr metric)

Electrically counterpoised dust (ECD) configurations (Papapetrou-Majundar class)
- static solutions (including charged discs)
- continuous black hole limit (extreme Reissner-Nordström metric) [Meinel and Hütten, 2011]

Getting an new exact solution with a good physical interpretation maybe with a continuous black hole limit (extreme Kerr-Newmann metric)
The boundary value problem for a rigidly rotating disc of charged dust

Motivation

Conventions

\[ \text{Metrik} \]

\[ g_{ab} \]: Signatur \((+,+,+,-)\)

Indices \((a, b, \ldots)\): run from 1 to 4

\[ \nabla \]: Operator: used like in cylindrical coordinates

Units: \(G = c = 1\) combined with Gauss system

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Conventions

- **Metrik** $g_{ab}$: Signatur $(+,+,+,−)$
Conventions

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The boundary value problem for a rigidly rotating disc of charged dust

Motivation

Conventions

- **Metrik $g_{ab}$**: Signatur $(+, +, +, -)$
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Conventions

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Table of Contents

1 Motivation

2 Basic concepts
   • The Einstein-Maxwell equations
   • The Model of Matter
   • Metric and Four-potential
   • Wave and Field equations
   • The Corotating frame

3 The boundary conditions
   • The disc case
   • Calculation
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The boundary value problem for a rigidly rotating disc of charged dust

Basic concepts

The Einstein-Maxwell equations

Einstein equations:

\[ R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab} \]  

Maxwell equations:

\[ F_{[ab};c] = 0 \] and \[ F_{ab};b = 4\pi j_a \]  

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Basic concepts

The Einstein-Maxwell equations

- **Einstein equations:**

  \[ R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab} \]  
  \[ (1) \]

- **Maxwell equations:**

  \[ F_{[ab;c]} = 0 \quad \text{and} \quad F^{ab} ;_b = 4\pi j^a \]  
  \[ (2) \]
Basic concepts

The Einstein-Maxwell equations

Field tensor $F_{ab}$ is antisymmetric

Wave equations:

$$1\sqrt{-g}\left[\sqrt{-g} g^{am} g^{bn}\left(A^n,m - A^m,n\right)\right]_{,b} = 4\pi j^a$$

Continuity equation:

$$j^a;_a = 0$$

Lorentz gauge:

$$A^a;_a = 1\sqrt{-g}\left(A^a\sqrt{-g}\right)_{,a} = 0$$

Four-potential $A^a$:

$$F_{ab} = A_{b;_a} - A_{a;_b} = A_{b,a} - A_{a,b}$$
The boundary value problem for a rigidly rotating disc of charged dust

Basic concepts
The Einstein-Maxwell equations

- **Four-potential** $A^a$:

  \[ F_{ab} = A_{b;a} - A_{a;b} = A_{b,a} - A_{a,b} \]  
  \hspace{1cm} (3)

- **Field tensor** $F_{ab}$ is antisymmetric
The boundary value problem for a rigidly rotating disc of charged dust

Basic concepts

The Einstein-Maxwell equations

Four-potential $A^a$:

$$F_{ab} = A_{b;a} - A_{a;b} = A_{b,a} - A_{a,b}$$  \hspace{1cm} (3)

Field tensor $F_{ab}$ is antisymmetric

Wave equations:

$$\frac{1}{\sqrt{-g}} \left[ \sqrt{-g} g^{am} g^{bn} (A_{n,m} - A_{m,n}) \right]_{,b} = 4\pi j^a$$  \hspace{1cm} (4)
Four-potential $A^a$: \[ F_{ab} = A_{b;a} - A_{a;b} = A_{b,a} - A_{a,b} \] (3)

Field tensor $F_{ab}$ is antisymmetric

Wave equations: 
\[ \frac{1}{\sqrt{-g}} \left[ \sqrt{-g} g^{am} g^{bn} (A_{n,m} - A_{m,n}) \right]_{,b} = 4\pi j^a \] (4)

Continuity equation: $j^a;_a = 0$
The boundary value problem for a rigidly rotating disc of charged dust

Basic concepts

The Einstein-Maxwell equations

Four-potential $A^a$:

$$ F_{ab} = A_b{};a - A_a{};b = A_{b,a} - A_{a,b} $$

Field tensor $F_{ab}$ is antisymmetric

Wave equations:

$$ \frac{1}{\sqrt{-g}} \left[ \sqrt{-g} g^{am} g^{bn} (A_{n,m} - A_{m,n}) \right]_{,b} = 4\pi j^a $$

Continuity equation: $j^a{};a = 0$

Lorentz gauge:

$$ A^a{};a = \frac{1}{\sqrt{-g}} \left( A^a \sqrt{-g} \right)_{,a} = 0 $$
Table of Contents

1 Motivation

2 Basic concepts
   - The Einstein-Maxwell equations
   - The Model of Matter
   - Metric and Four-potential
   - Wave and Field equations
   - The Corotating frame

3 The boundary conditions
   - The disc case
   - Calculation
   - Physical interpretation
Isolated charged dust configuration in equilibrium
Isolated charged dust configuration in equilibrium

- Far field:
Isolated charged dust configuration in equilibrium

- Far field: $g_{ab} = \eta_{ab} + \mathcal{O}(r^{-1})$ (see also [Stephani, 1991])
Isolated charged dust configuration in equilibrium

- **Far field**: \( g_{ab} = \eta_{ab} + \mathcal{O}(r^{-1}) \) (see also [Stephani, 1991])
  \( \rightarrow \) asymptotically flat spacetime
Isolated charged dust configuration in equilibrium

- **Far field:** $g_{ab} = \eta_{ab} + \mathcal{O}(r^{-1})$ (see also [Stephani, 1991])
  → asymptotically flat spacetime

- **Stationarity and axisymmetry:**
Isolated charged dust configuration in equilibrium

- **Far field:** \( g_{ab} = \eta_{ab} + \mathcal{O}(r^{-1}) \) (see also [Stephani, 1991])
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- **Stationarity and axisymmetry:**
  \( \rightarrow \) two Killing vectors \( \xi \) and \( \eta \)
Isolated charged dust configuration in equilibrium

- **Far field**: $g_{ab} = \eta_{ab} + \mathcal{O}(r^{-1})$  
  (see also [Stephani, 1991])
  → asymptotically flat spacetime

- **Stationarity and axisymmetry**: 
  → two Killing vectors $\xi$ and $\eta$

- **Electrovacuum**: $T_{ab}^{\text{em}} = \frac{1}{4\pi} \left( F_{ac} F_{b}^{\ c} - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right)$
Isolated charged dust configuration in equilibrium

- **Far field:** $g_{ab} = \eta_{ab} + \mathcal{O}(r^{-1})$ (see also [Stephani, 1991])
  $\rightarrow$ asymptotically flat spacetime
- **Stationarity and axisymmetry:**
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- **Electrovacuum:** $T_{ab}^{em} = \frac{1}{4\pi} \left( F_{ac}F_c^b - \frac{1}{4} g_{ab} F_{cd}F^{cd} \right)$
- **Charged dust:**
  $$T_{ab} = \mu u_a u_b + T_{ab}^{em}$$  \hspace{1cm} (6)
Isolated charged dust configuration in equilibrium

- Far field: $g_{ab} = \eta_{ab} + \mathcal{O}(r^{-1})$ (see also [Stephani, 1991])
  $\rightarrow$ asymptotically flat spacetime

- Stationarity and axisymmetry:
  $\rightarrow$ two Killing vectors $\xi$ and $\eta$

- Electrovacuum: $T^{\text{em}}_{ab} = \frac{1}{4\pi} \left( F_{ac} F^c_b - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right)$

- Charged dust:
  \[ T_{ab} = \mu u_a u_b + T^{\text{em}}_{ab} \]  \hspace{1cm} (6)

  $\rightarrow T = -\mu$
Isolated charged dust configuration in equilibrium

- **Far field**: $g_{ab} = \eta_{ab} + O(r^{-1})$ (see also [Stephani, 1991])
  → asymptotically flat spacetime

- **Stationarity and axisymmetry**: 
  → two Killing vectors $\xi$ and $\eta$

- **Electrovacuum**: $T_{ab}^{\text{em}} = \frac{1}{4\pi} \left( F_{ac}F_{cb} - \frac{1}{4} g_{ab}F_{cd}F^{cd} \right)$

- **Charged dust**: 
  $$T_{ab} = \mu u_a u_b + T_{ab}^{\text{em}}$$  \hspace{1cm} (6)
  → $T = -\mu$

- **Convective current of particles**: $j^a = \rho_{\text{el}} u^a$
Isolated charged dust configuration in equilibrium

- **Far field**: $g_{ab} = \eta_{ab} + \mathcal{O}(r^{-1})$ (see also [Stephani, 1991])
  \[ \rightarrow \text{asymptotically flat spacetime} \]

- **Stationarity and axisymmetry**: \[ \rightarrow \text{two Killing vectors } \xi \text{ and } \eta \]

- **Electrovacuum**: $T_{ab}^{\text{em}} = \frac{1}{4\pi} \left( F_{ac} F_b^c - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right)$

- **Charged dust**:
  \[ T_{ab} = \mu u_a u_b + T_{ab}^{\text{em}} \]
  \[ \rightarrow T = -\mu \]

- **Convective current of particles**: $j^a = \varrho_{\text{el}} u^a$

- **Charge density is proportional to mass density**: $\varrho_{\text{el}} = \epsilon \mu$
  with constant $\epsilon \in [-1, 1]$
Isolated charged dust configuration in equilibrium

- Far field: $g_{ab} = \eta_{ab} + O(r^{-1})$ (see also [Stephani, 1991])
  $\rightarrow$ asymptotically flat spacetime
- Stationarity and axisymmetry:
  $\rightarrow$ two Killing vectors $\xi$ and $\eta$
- Electrovacuum: $T_{ab}^{\text{em}} = \frac{1}{4\pi} \left( F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right)$
- Charged dust:
  \[
  T_{ab} = \mu u_a u_b + T_{ab}^{\text{em}}
  \]  
  $\rightarrow T = -\mu$
- Convective current of particles: $j^a = \rho_{el} u^a$
- Charge density is proportional to mass density: $\rho_{el} = \epsilon \mu$
  with constant $\epsilon \in [-1, 1]$
- $\epsilon = 0 \rightarrow$ uncharged case
The boundary value problem for a rigidly rotating disc of charged dust

Basic concepts

The Model of Matter

Isolated charged dust configuration in equilibrium

- Far field: \( g_{ab} = \eta_{ab} + O\left(r^{-1}\right) \)  (see also [Stephani, 1991])
  \( \rightarrow \) asymptotically flat spacetime
- Stationarity and axisymmetry:
  \( \rightarrow \) two Killing vectors \( \xi \) and \( \eta \)
- Electrovacuum: \( T_{ab}^{\text{em}} = \frac{1}{4\pi} \left( F_{ac}F_{b}^{\ c} - \frac{1}{4}g_{ab}F_{cd}F^{cd} \right) \)
- Charged dust:
  \( T_{ab} = \mu u_{a}u_{b} + T_{ab}^{\text{em}} \)  \hspace{1cm} (6)
  \( \rightarrow T = -\mu \)
- Convective current of particles: \( j^{a} = \rho_{\text{el}}u^{a} \)
- Charge density is proportional to mass density: \( \rho_{\text{el}} = \epsilon \mu \)
  with constant \( \epsilon \in [-1, 1] \)
- \( \epsilon = 0 \rightarrow \) uncharged case
- \( \epsilon = \pm 1 \rightarrow \) ECD case
Local energy-momentum conservation: $T^{ab}_{;b} = 0$
The boundary value problem for a rigidly rotating disc of charged dust

Basic concepts

The Model of Matter

- Local energy-momentum conservation: $T^{ab}_{\;;b} = 0$
- Baryonic mass balance:

$$\left(\mu u^a\right)_{;a} = 0 \Rightarrow \left(\epsilon\mu u^a\right)_{;a} = j^a_{;a} = 0$$
Local energy-momentum conservation: $T^{ab}_{;b} = 0$

Baryonic mass balance:

$$(\mu u^a)_{;a} = 0 \Rightarrow (\epsilon \mu u^a)_{;a} = j^a_{;a} = 0$$

Equations of motion:

$$\mu \dot{u}^a = \mu \frac{Du^a}{D\tau} = f^a$$
The boundary value problem for a rigidly rotating disc of charged dust

Basic concepts

The Model of Matter

- Local energy-momentum conservation: $T^{ab}_{;b} = 0$
- Baryonic mass balance:
  
  $$(\mu u^a)_{;a} = 0 \Rightarrow (\epsilon \mu u^a)_{;a} = j^a_{;a} = 0$$

- Equations of motion:
  
  $$\mu \dot{u}^a = \mu \frac{Du^a}{D\tau} = f^a$$

  with Lorentz force density $f^a = \epsilon \mu F^{ab} u_b$
The boundary value problem for a rigidly rotating disc of charged dust

Basic concepts

The Model of Matter

- Local energy-momentum conservation: $T^{ab}_{\ ;b} = 0$
- Baryonic mass balance:
  \[(\mu u^a)_{;a} = 0 \implies (\epsilon \mu u^a)_{;a} = j^a_{;a} = 0\]

- Equations of motion:
  \[
  \mu \dot{u}^a = \mu \frac{D u^a}{D\tau} = f^a
  \]
  with Lorentz force density $f^a = \epsilon \mu F^{ab} u_b$
  → Local mass conservation and the particles are moving under the influence of the Lorentz force
Table of Contents

1 Motivation

2 Basic concepts
   - The Einstein-Maxwell equations
   - The Model of Matter
   - Metric and Four-potential
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3 The boundary conditions
   - The disc case
   - Calculation
   - Physical interpretation
Coordinate system with $\xi = \partial_t$ and $\eta = \partial_\phi$
Coordinate system with $\xi = \partial_t$ and $\eta = \partial_\phi$
→ Lewis-Papapetrou Coordinates ($\rho, \zeta, \varphi, t$)
Boundary value problem for a rigidly rotating disc of charged dust

**Basic concepts**

- Metric and Four-potential

  - Coordinate system with $\xi = \partial_t$ and $\eta = \partial_\varphi$
    → Lewis-Papapetrou Coordinates $(\rho, \zeta, \varphi, t)$

  - Metric

    \[
    ds^2 = f^{-1} \left[ h (d\rho^2 + d\zeta^2) + W^2 d\varphi^2 \right] - f (dt + a d\varphi)^2
    \]

    with four metric potentials $f = e^{2U}$, $h = e^{2k}$, $W$ and $a$ only depending on $\rho$ and $\zeta$
Coordinate system with $\xi = \partial_t$ and $\eta = \partial_\varphi$  
→ Lewis-Papapetrou Coordinates $(\rho, \zeta, \varphi, t)$

Metric

$$ds^2 = f^{-1} \left[ h (d\rho^2 + d\zeta^2) + W^2 d\varphi^2 \right] - f (dt + ad\varphi)^2$$  \hspace{1cm} (7)

with four metric potentials $f = e^{2U}$, $h = e^{2k}$, $W$ and $a$ only depending on $\rho$ and $\zeta$

Circularity condition
Coordinate system with $\xi = \partial_t$ and $\eta = \partial_\varphi$ → Lewis-Papapetrou Coordinates $(\varrho, \varsigma, \varphi, t)$

Metric

$$ds^2 = f^{-1} \left[ h (d\varrho^2 + d\varsigma^2) + W^2 d\varphi^2 \right] - f (dt + ad\varphi)^2$$  \hspace{1cm} (7)

with four metric potentials $f = e^{2U}, h = e^{2k}, W$ and $a$ only depending on $\varrho$ and $\varsigma$

Circularity condition → $u^a = (0, 0, u^\varphi, u^t), \quad j^a = (0, 0, j^\varphi, j^t)$

The boundary value problem for a rigidly rotating disc of charged dust
Coordinate system with $\xi = \partial_t$ and $\eta = \partial_\varphi$
$\rightarrow$ Lewis-Papapetrou Coordinates $(\varrho, \zeta, \varphi, t)$

Metric

$$ds^2 = f^{-1} \left[ h \left( d\varrho^2 + d\zeta^2 \right) + W^2 d\varphi^2 \right] - f \left( dt + a d\varphi \right)^2$$  \hspace{1cm} (7)

with four metric potentials $f = e^{2U}, h = e^{2k}, W$ and $a$ only depending on $\varrho$ and $\zeta$

Circularity condition $\rightarrow u^a = (0, 0, u^\varphi, u^t), \quad j^a = (0, 0, j^\varphi, j^t)$
$\rightarrow$ We can use the metric (7) \textit{inside} the matter
Coordinate system with $\xi = \partial_t$ and $\eta = \partial_\varphi$
→ Lewis-Papapetrou Coordinates $(\varrho, \zeta, \varphi, t)$

Metric

$$ds^2 = f^{-1} \left[ h (d\varrho^2 + d\zeta^2) + W^2 d\varphi^2 \right] - f (dt + ad\varphi)^2 \quad (7)$$

with four metric potentials $f = e^{2U}$, $h = e^{2k}$, $W$ and $a$ only depending on $\varrho$ and $\zeta$

Circularity condition → $u^a = (0, 0, u^\varphi, u^t)$, $j^a = (0, 0, j^\varphi, j^t)$
→ We can use the metric (7) inside the matter

$a$: Gravitomagnetic potential
The boundary value problem for a rigidly rotating disc of charged dust

- **Basic concepts**
- **Metric and Four-potential**

- Coordinate system with \( \xi = \partial_t \) and \( \eta = \partial_\varphi \)
  \( \rightarrow \) Lewis-Papapetrou Coordinates \( (\varrho, \zeta, \varphi, t) \)

- Metric

\[
ds^2 = f^{-1} \left[ h \left( d\varrho^2 + d\zeta^2 \right) + W^2 d\varphi^2 \right] - f \left( dt + a d\varphi \right)^2 \tag{7}
\]

with four metric potentials \( f = e^{2U}, h = e^{2k}, W \) and \( a \) only depending on \( \varrho \) and \( \zeta \)

- Circularity condition \( \rightarrow u^a = (0, 0, u^\varphi, u^t), \ j^a = (0, 0, j^\varphi, j^t) \)
  \( \rightarrow \) We can use the metric (7) *inside* the matter

- **a**: Gravitomagnetic potential, newtonian limit: \( U \rightarrow U^G \)
The boundary value problem for a rigidly rotating disc of charged dust

Basic concepts

Metric and Four-potential

- Coordinate system with $\xi = \partial_t$ and $\eta = \partial_\varphi$
  $\rightarrow$ Lewis-Papapetrou Coordinates $(\varrho, \zeta, \varphi, t)$

- Metric

$$ds^2 = f^{-1} [h (d\varrho^2 + d\zeta^2) + W^2 d\varphi^2] - f (dt + ad\varphi)^2$$  \hspace{1cm} (7)

  with four metric potentials $f = e^{2U}$, $h = e^{2k}$, $W$ and $a$ only depending on $\varrho$ and $\zeta$

- Circularity condition $\rightarrow u^a = (0, 0, u^\varphi, u^t)$, $j^a = (0, 0, j^\varphi, j^t)$
  $\rightarrow$ We can use the metric (7) inside the matter

- $a$: Gravitomagnetic potential, newtonian limit: $U \rightarrow U^G$

- Rigidly rotating: The matter rotates with the constant angular velocity $\Omega = d\varphi/dt$
The boundary value problem for a rigidly rotating disc of charged dust

- **Basic concepts**
- **Metric and Four-potential**

- Coordinate system with $\xi = \partial_t$ and $\eta = \partial_\varphi$
  → Lewis-Papapetrou Coordinates $(\varrho, \zeta, \varphi, t)$

- Metric

$$ds^2 = f^{-1} \left[ h (d\varrho^2 + d\zeta^2) + W^2 d\varphi^2 \right] - f (dt + a d\varphi)^2 \quad (7)$$

with four metric potentials $f = e^{2U}$, $h = e^{2k}$, $W$ and $a$ only depending on $\varrho$ and $\zeta$

- Circularity condition → $u^a = (0, 0, u^\varphi, u^t)$, $j^a = (0, 0, j^\varphi, j^t)$
  → We can use the metric (7) inside the matter

- $a$: Gravitomagnetic potential, newtonian limit: $U \to U^G$

- Rigidly rotating: The matter rotates with the constant angular velocity $\Omega = d\varphi/dt$

- Far field: $ds^2 \to$ Minkowski metric in cylindrical coordinates
The boundary value problem for a rigidly rotating disc of charged dust

**Basic concepts**

**Metric and Four-potential**

- Coordinate system with \( \xi = \partial_t \) and \( \eta = \partial_\varphi \)
  \( \rightarrow \) Lewis-Papapetrou Coordinates \((\varrho, \zeta, \varphi, t)\)

- Metric

\[
\text{d}s^2 = f^{-1} \left[ h \left( \text{d}\varrho^2 + \text{d}\zeta^2 \right) + W^2 \text{d}\varphi^2 \right] - f \left( \text{d}t + a \text{d}\varphi \right)^2
\]  

\(7\)

with four metric potentials \( f = e^{2U} \), \( h = e^{2k} \), \( W \) and \( a \) only depending on \( \varrho \) and \( \zeta \)

- Circularity condition \( \rightarrow u^a = (0, 0, u^\varphi, u^t), \ j^a = (0, 0, j^\varphi, j^t) \)
  \( \rightarrow \) We can use the metric \((7)\) inside the matter

- \( a \): Gravitomagnetic potential, newtonian limit: \( U \rightarrow U^G \)

- Rigidly rotating: The matter rotates with the constant angular velocity \( \Omega = \frac{\text{d}\varphi}{\text{d}t} \)

- Far field: \( \text{d}s^2 \rightarrow \) Minkowski metric in cylindrical coordinates

\( U \rightarrow 0, a \rightarrow 0, k \rightarrow 0 \) and \( W \rightarrow \varrho \)
Coordinate system with $\xi = \partial_t$ and $\eta = \partial_\varphi$
 → Lewis-Papapetrou Coordinates $(\varrho, \zeta, \varphi, t)$

Metric

\[
ds^2 = f^{-1} \left[ h \left( d\varrho^2 + d\zeta^2 \right) + W^2 d\varphi^2 \right] - f \left( dt + a d\varphi \right)^2 \tag{7}\]

with four metric potentials $f = e^{2U}$, $h = e^{2k}$, $W$ and $a$ only depending on $\varrho$ and $\zeta$

Circularity condition → $u^a = (0, 0, u^\varphi, u^t)$, $j^a = (0, 0, j^\varphi, j^t)$
 → We can use the metric (7) inside the matter

$a$: Gravitomagnetic potential, newtonian limit: $U \to U^G$

Rigidly rotating: The matter rotates with the constant angular velocity $\Omega = d\varphi/dt$

Far field: $ds^2 \to$ Minkowski metric in cylindrical coordinates $U \to 0, a \to 0, k \to 0$ and $W \to \varrho$

Lorentz gauge → $A_a = (0, 0, A^\varphi, A^t)$, $A^a = (0, 0, A^\varphi, A^t)$
Global field equation: \( W,\varrho,\varrho + W,\zeta,\zeta = 0 \)
Global field equation: $W,\varrho,\varrho + W,\zeta,\zeta = 0$

asymptotic behaviour $\to W = \varrho$
Global field equation: $W,\varrho,\varrho + W,\zeta,\zeta = 0$

asymptotic behaviour $\to W = \varrho$

Lewis-Papapetrou-Weyl Coordinates:

$$ds^2 = f^{-1} \left[ h (d\varrho^2 + d\zeta^2) + \varrho^2 d\varphi^2 \right] - f (dt + a d\varphi)^2$$
Global field equation: \( W,\varrho,\varrho + W,\zeta,\zeta = 0 \)

asymptotic behaviour \( \rightarrow W = \varrho \)

Lewis-Papapetrou-Weyl Coordinates:

\[
\begin{align*}
ds^2 &= f^{-1} \left[ h \left( d\varrho^2 + d\zeta^2 \right) + \varrho^2 d\varphi^2 \right] - f \left( dt + ad\varphi \right)^2
\end{align*}
\]

Reflectional symmetry:

\( \rightarrow g_{ab}(\varrho, -\zeta) = g_{ab}(\varrho, \zeta) \)
The boundary value problem for a rigidly rotating disc of charged dust

- Basic concepts
- Metric and Four-potential

- Global field equation: \( W_{,\varrho,\varrho} + W_{,\zeta,\zeta} = 0 \)
  asymptotic behaviour \( \rightarrow W = \varrho \)

- Lewis-Papapetrou-Weyl Coordinates:
  \[
  ds^2 = f^{-1} \left[ h \left( d\varrho^2 + d\zeta^2 \right) + \varrho^2 d\varphi^2 \right] - f \left( dt + a d\varphi \right)^2
  \]

- Reflectional symmetry:
  \( \rightarrow g_{ab}(\varrho, -\zeta) = g_{ab}(\varrho, \zeta) \) and \( A^a(\varrho, -\zeta) = A^a(\varrho, \zeta) \)
Global field equation: $W_{,\varrho,\varrho} + W_{,\zeta,\zeta} = 0$

asymptotic behaviour $\rightarrow W = \varrho$

Lewis-Papapetrou-Weyl Coordinates:

$$ds^2 = f^{-1} \left[ h \left( d\varrho^2 + d\zeta^2 \right) + \varrho^2 d\varphi^2 \right] - f \left( dt + a d\varphi \right)^2$$

Reflectional symmetry:

$g_{ab}(\varrho, -\zeta) = g_{ab}(\varrho, \zeta)$ and $A^a(\varrho, -\zeta) = A^a(\varrho, \zeta)$

Potentials: $f(\varrho, -\zeta) = f(\varrho, \zeta)$, ...
# Table of Contents

1 Motivation

2 Basic concepts
   - The Einstein-Maxwell equations
   - The Model of Matter
   - Metric and Four-potential
   - Wave and Field equations
   - The Corotating frame

3 The boundary conditions
   - The disc case
   - Calculation
   - Physical interpretation
The boundary value problem for a rigidly rotating disc of charged dust

- **Basic concepts**

- **Wave and Field equations**

- Wave and field equations:

\[ 4\pi\varepsilon\mu f^{-1}hu^\varphi = \nabla \cdot \left[ \frac{f}{\varrho^2} (a\nabla A_t - \nabla A_\varphi) \right] \]  

\[ -4\pi\varepsilon\mu f^{-1}hu^t = \nabla \cdot \left[ \frac{af}{\varrho^2} (a\nabla A_t - \nabla A_\varphi) - \frac{1}{f} \nabla A_t \right] \]  

\[ 16\pi\mu\varrho f^{-1}hu^\varphi (\varepsilon A_t + u_t) = \nabla \cdot \left[ \frac{f^2}{\varrho^2} \nabla a + 4\frac{f}{\varrho^2} A_t (a\nabla A_t - \nabla A_\varphi) \right] \]  

\[ 8\pi\mu fh [2u^\varphi (u_\varphi - au_t) + 1] = f\Delta f - (\nabla f)^2 + \frac{f^4}{\varrho^2} (\nabla a)^2 \]

\[ - 2f \left[ (\nabla A_t)^2 + \frac{f^2}{\varrho^2} (a\nabla A_t - \nabla A_\varphi)^2 \right] \]
Wave and field equations:

\[ 4\pi \epsilon \mu f^{-1} hu^\varphi = \nabla \cdot \left[ \frac{f}{\varrho^2} (a \nabla A_t - \nabla A_\varphi) \right] \] (8)

\[ -4\pi \epsilon \mu f^{-1} hu^t = \nabla \cdot \left[ \frac{af}{\varrho^2} (a \nabla A_t - \nabla A_\varphi) - \frac{1}{f} \nabla A_t \right] \] (9)

\[ 16\pi \mu \varrho f^{-1} hu^\varphi (\epsilon A_t + u_t) = \nabla \cdot \left[ \frac{f^2}{\varrho^2} \nabla a + 4 \frac{f}{\varrho^2} A_t (a \nabla A_t - \nabla A_\varphi) \right] \] (10)

\[ 8\pi \mu fh \left[ 2u^\varphi (u_\varphi - au_t) + 1 \right] = f \Delta f - (\nabla f)^2 + \frac{f^4}{\varrho^2} (\nabla a)^2 \]

\[ -2f \left[ (\nabla A_t)^2 + \frac{f^2}{\varrho^2} (a \nabla A_t - \nabla A_\varphi)^2 \right] \] (11)

Complex Ernst potentials in electrovacuum \((\mu = 0)\):
The boundary value problem for a rigidly rotating disc of charged dust

- **Basic concepts**
- **Wave and Field equations**

**Wave and field equations:**

\[
4\pi\varepsilon\mu f^{-1}hu^\varphi = \nabla \cdot \left[ \frac{f}{\varrho^2} (a\nabla A_t - \nabla A_\varphi) \right] \quad (8)
\]

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-4\pi\varepsilon\mu f^{-1}hu^t = \nabla \cdot \left[ \frac{af}{\varrho^2} (a\nabla A_t - \nabla A_\varphi) - \frac{1}{f} \nabla A_t \right] \quad (9)
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\[
16\pi\mu\varrho f^{-1}hu^\varphi (\varepsilon A_t + u_t) = \nabla \cdot \left[ \frac{f^2}{\varrho^2} \nabla a + 4\frac{f}{\varrho^2} A_t (a\nabla A_t - \nabla A_\varphi) \right] \quad (10)
\]

\[
8\pi\mu fh [2u^\varphi (u_\varphi - au_t) + 1] = f\Delta f - (\nabla f)^2 + \frac{f^4}{\varrho^2} (\nabla a)^2 - 2f \left[ (\nabla A_t)^2 + \frac{f^2}{\varrho^2} (a\nabla A_t - \nabla A_\varphi)^2 \right] \quad (11)
\]

**Complex Ernst potentials in electrovacuum (\(\mu = 0\)):**

- \(\Phi = \alpha + i\beta\) with \(\alpha = -A_t\) and

\[
\nabla \times \left( \frac{\beta}{\varrho} \mathbf{e}_\varphi \right) = -\frac{f}{\varrho^2} (a\nabla A_t - \nabla A_\varphi)
\]
The boundary value problem for a rigidly rotating disc of charged dust

Basic concepts

Wave and Field equations

Wave and field equations:

\[ 4\pi\epsilon \mu f^{-1} hu^\varphi = \nabla \cdot \left[ \frac{f}{\varrho^2} (a\nabla A_t - \nabla A_\varphi) \right] \]  

(8)

\[ -4\pi\epsilon \mu f^{-1} hu^t = \nabla \cdot \left[ \frac{af}{\varrho^2} (a\nabla A_t - \nabla A_\varphi) - \frac{1}{f} \nabla A_t \right] \]  

(9)

\[ 16\pi\mu\varrho f^{-1} hu^\varphi (\epsilon A_t + u_t) = \nabla \cdot \left[ \frac{f^2}{\varrho^2} \nabla a + 4\frac{f}{\varrho^2} A_t (a\nabla A_t - \nabla A_\varphi) \right] \]  

(10)

\[ 8\pi\mu fh [2u^\varphi (u_\varphi - au_t) + 1] = f\Delta f - (\nabla f)^2 + \frac{f^4}{\varrho^2} (\nabla a)^2 \]

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(11)

Complex Ernst potentials in electrovacuum \((\mu = 0)\):

- \(\Phi = \alpha + i\beta\) with \(\alpha = -A_t\) and

\[ \nabla \times \left( \frac{\beta}{\varrho} e_\varphi \right) = -\frac{f}{\varrho^2} (a\nabla A_t - \nabla A_\varphi) \]

- \(E = (f - \bar{\Phi}\Phi) + ib\) with

\[ \nabla \times \left( \frac{b}{\varrho} e_\varphi \right) = -\frac{f^2}{\varrho^2} \nabla a - 2 \Im \left[ \bar{\Phi} \nabla \times \left( \frac{\Phi}{\varrho} e_\varphi \right) \right] \]
The boundary value problem for a rigidly rotating disc of charged dust

- **Basic concepts**

- **Wave and Field equations**

  - Ernst equations [Ernst, 1968]:

    \[
    (\Re E + \Phi\Phi) \Delta E = (\nabla E + 2\Phi \nabla \Phi) \cdot \nabla E \tag{12}
    \]

    \[
    (\Re E + \Phi\Phi) \Delta \Phi = (\nabla E + 2\Phi \nabla \Phi) \cdot \nabla \Phi \tag{13}
    \]
The boundary value problem for a rigidly rotating disc of charged dust

### Basic concepts

### Wave and Field equations

- **Ernst equations** [Ernst, 1968]:
  \[
  (\mathcal{R}\mathcal{E} + \bar{\Phi}\Phi) \Delta \mathcal{E} = (\nabla \mathcal{E} + 2\bar{\Phi}\nabla \Phi) \cdot \nabla \mathcal{E} \quad (12)
  \]
  \[
  (\mathcal{R}\mathcal{E} + \bar{\Phi}\Phi) \Delta \Phi = (\nabla \mathcal{E} + 2\bar{\Phi}\nabla \Phi) \cdot \nabla \Phi \quad (13)
  \]

- **Field equations for** \( h \):\
  \[
  (\ln h)_{,\rho} = \frac{1}{2} \rho \left( (\ln f)^2_{,\rho} - (\ln f)^2_{,\zeta} - \frac{f^2}{\rho^2} (\alpha^2_{,\rho} - \alpha^2_{,\zeta}) \right)
  
  + 2 \left[ \frac{f}{\rho} (A^2_{,\rho} - A^2_{,\zeta}) - \frac{\rho^2 - \alpha^2 f^2}{f \rho} (A^2_{t,\rho} - A^2_{t,\zeta}) - 2 \frac{af}{\rho} (A_{\varphi,\rho} A_{t,\rho} - A_{\varphi,\zeta} A_{t,\zeta}) \right]
  \]

  \[
  (\ln h)_{,\zeta} = \rho \left( (\ln f)_{,\rho} (\ln f)_{,\zeta} - \frac{f^2}{\rho^2} \alpha_{,\rho} \alpha_{,\zeta} \right)
  
  + 4 \left[ \frac{f}{\rho} A_{\varphi,\rho} A_{\varphi,\zeta} - \frac{\rho^2 - \alpha^2 f^2}{f \rho} A_{t,\rho} A_{t,\zeta} - \frac{af}{\rho} (A_{\varphi,\zeta} A_{t,\rho} + A_{\varphi,\rho} A_{t,\zeta}) \right]
  \]

  → In electrovacuum \( h \) can be calculated via a path-independent line integral
Table of Contents

1 Motivation

2 Basic concepts
   - The Einstein-Maxwell equations
   - The Model of Matter
   - Metric and Four-potential
   - Wave and Field equations
   - The Corotating frame

3 The boundary conditions
   - The disc case
   - Calculation
   - Physical interpretation
The boundary value problem for a rigidly rotating disc of charged dust

Basic concepts

The Co-rotating frame

Coordinate transformation:

\[ \rho' = \rho, \quad \zeta' = \zeta, \quad \varphi' = \varphi - \Omega t, \quad t' = t \]
The boundary value problem for a rigidly rotating disc of charged dust

### Basic concepts

#### The Corotating frame

- **Coordinate transformation:**
  \[ \rho' = \rho, \quad \zeta' = \zeta, \quad \varphi' = \varphi - \Omega t, \quad t' = t \]

- **Metric retains its form:**
  \[ ds'^2 = f'^{-1} \left[ h' \left( d\rho'^2 + d\zeta'^2 \right) + \rho'^2 d\varphi'^2 \right] - f' \left( dt' + a'd\varphi' \right)^2 \]

  with:

  \[ f' = f \left[ (1 + \Omega a)^2 - \frac{\Omega^2 \rho^2}{f^2} \right] \]

  \[ (1 - \Omega a') f' = (1 + \Omega a) f \]

  \[ f'^{-1} h' = f^{-1} h \]
The boundary value problem for a rigidly rotating disc of charged dust

Basic concepts

The Corotating frame

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- Four-potential: \( A_{\varphi'} = A_\varphi \) and \( A_{t'} = A_t + \Omega A_\varphi \)
The boundary value problem for a rigidly rotating disc of charged dust

Basic concepts

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- Four-velocity: \( u^{a'} = \left( 0, 0, 0, e^{-U'} \right) \)
Coordinate transformation:

\[ \rho' = \rho, \quad \zeta' = \zeta, \quad \varphi' = \varphi - \Omega t, \quad t' = t \]

Metric retains its form:

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with:

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Wave and field equations retaining their form
Basic concepts

The Corotating frame

- Coordinate transformation:
  \[ \rho' = \rho, \quad \zeta' = \zeta, \quad \varphi' = \varphi - \Omega t, \quad t' = t \]

- Metric retains its form:
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- Four-potential: \( A_{\varphi'} = A_\varphi \) and \( A_{t'} = A_t + \Omega A_\varphi \)
- Four-velocity: \( u^{a'} = \left( 0, 0, 0, e^{-U'} \right) \)
- Wave and field equations retaining their form
- The metric potentials have a different asymptotic behaviour
# Table of Contents

1. **Motivation**

2. **Basic concepts**
   - The Einstein-Maxwell equations
   - The Model of Matter
   - Metric and Four-potential
   - Wave and Field equations
   - The Corotating frame

3. **The boundary conditions**
   - The disc case
   - Calculation
   - Physical interpretation
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

The disc case

- Assuming a circular disc with coordinate radius $\rho_0$
Assuming a circular disc with coordinate radius $\varrho_0$

→ domain to be considered as a world cylinder of the surface elements of the two-dimensional surface $\Sigma_2$. 

The boundary conditions

The disc case
Assuming a circular disc with coordinate radius $\varrho_0$

→ domain to be considered as a world cylinder of the surface elements of the two-dimensional surface $\Sigma_2$,

$$\Sigma_2 : \quad \zeta = 0 \quad (0 \leq \varrho \leq \varrho_0), \quad t = \text{constant}$$
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

The disc case

Assuming a circular disc with coordinate radius \( \varrho_0 \)
→ domain to be considered as a world cylinder of the surface elements of the two-dimensional surface \( \Sigma_2 \),

\[
\Sigma_2 : \quad \zeta = 0 \quad (0 \leq \varrho \leq \varrho_0), \quad t = \text{constant}
\]

Mass density: \( \mu = \sigma (\varrho) e^{2U - 2k} \delta (\zeta) \)
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

The disc case

Assuming a circular disc with coordinate radius $\rho_0$

→ domain to be considered as a world cylinder of the surface elements of the two-dimensional surface $\Sigma_2$,

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$\sigma(\rho)$: coordinate dependend surface mass density
Assuming a circular disc with coordinate radius $\varrho_0$ → domain to be considered as a world cylinder of the surface elements of the two-dimensional surface $\Sigma_2$,

$$
\Sigma_2 : \quad \zeta = 0 \quad (0 \leq \varrho \leq \varrho_0), \quad t = \text{constant}
$$

Mass density: $\mu = \sigma (\varrho) e^{2U-2k} \delta (\zeta)$

$\sigma (\varrho)$: coordinate dependent surface mass density → invariant surface mass density $\sigma_p (\varrho) = e^{U-k} \sigma (\varrho)$
Assuming a circular disc with coordinate radius $\rho_0$ → domain to be considered as a world cylinder of the surface elements of the two-dimensional surface $\Sigma_2$,

$$\Sigma_2 : \quad \zeta = 0 \quad (0 \leq \rho \leq \rho_0), \quad t = \text{constant}$$

Mass density: $\mu = \sigma (\rho) e^{2U-2k}\delta (\zeta)$

$\sigma (\rho)$: coordinate dependend surface mass density → invariant surface mass density $\sigma_p (\rho) = e^{U-k}\sigma (\rho)$

Idea: Getting the *electrovacuum* solution from the Ernst equations with boundary data derived from the wave and field equations *inside* the disc

see also [Meinel et al., 2008]
Table of Contents

1 Motivation

2 Basic concepts
   - The Einstein-Maxwell equations
   - The Model of Matter
   - Metric and Four-potential
   - Wave and Field equations
   - The Corotating frame

3 The boundary conditions
   - The disc case
   - Calculation
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Martin Breithaupt
Research Training Group "Quantum and Gravitational Fields"
Asymptotic behaviour: $\Phi \to 0$ and $E \to 1$ as $\varrho^2 + \zeta^2 \to \infty$
The boundary value problem for a rigidly rotating disc of charged dust

Asymptotic behaviour: \( \Phi \to 0 \) and \( \mathcal{E} \to 1 \) as \( \rho^2 + \zeta^2 \to \infty \)

The boundary conditions on the disc are simpler in the corotating frame
Asymptotic behaviour: $\Phi \to 0$ and $\mathcal{E} \to 1$ as $\rho^2 + \zeta^2 \to \infty$

The boundary conditions on the disc are simpler in the corotating frame

Reflectional symmetry:
Results for the potentials hold in the corotating frame
Asymptotic behaviour: $\Phi \to 0$ and $\mathcal{E} \to 1$ as $\varrho^2 + \zeta^2 \to \infty$

The boundary conditions on the disc are simpler in the corotating frame.

Reflectional symmetry:
Results for the potentials hold in the corotating frame
$\rightarrow$ Potentials are continuous across the disc
Asymptotic behaviour: $\Phi \to 0$ and $E \to 1$ as $\rho^2 + \zeta^2 \to \infty$

The boundary conditions on the disc are simpler in the co-rotating frame

Reflectional symmetry:
Results for the potentials hold in the co-rotating frame
$\rightarrow$ Potentials are continuous across the disc
$\rightarrow$ Normal derivatives $f'_,\zeta'_{|\zeta'=0^+} = -f'_,\zeta'_{|\zeta'=0^-}$, ... may jump
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

Calculation

- Asymptotic behaviour: $\Phi \to 0$ and $E \to 1$ as $\varrho^2 + \zeta^2 \to \infty$
- The boundary conditions on the disc are simpler in the corotating frame
- Reflectional symmetry:
  Results for the potentials hold in the corotating frame
  → Potentials are continuous across the disc
  → Normal derivatives $f'_{,\zeta'} \bigg|_{\zeta' = 0^+} = -f'_{,\zeta'} \bigg|_{\zeta' = 0^-}$, ... may jump
  → Analogy to electrodynamics
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

Calculation

- Asymptotic behaviour: $\Phi \to 0$ and $E \to 1$ as $\varrho^2 + \zeta^2 \to \infty$

- The boundary conditions on the disc are simpler in the corotating frame

- Reflectional symmetry:
  - Results for the potentials hold in the corotating frame
  - Potentials are continuous across the disc
  - Normal derivatives $f',\zeta'|_{\zeta'=0^+} = -f',\zeta'|_{\zeta'=0^-}$, ... may jump
  - Analogy to electrodynamics
  - Integrating the field and wave equations over a small cylinder centered on the disc and applying Gauss theorem
Asymptotic behaviour: \( \Phi \to 0 \) and \( \mathcal{E} \to 1 \) as \( \varrho^2 + \zeta^2 \to \infty \)

The boundary conditions on the disc are simpler in the corotating frame.

Reflectional symmetry:
Results for the potentials hold in the corotating frame
→ Potentials are continuous across the disc
→ Normal derivatives \( f',\zeta' \big|_{\zeta'=0^+} = -f',\zeta' \big|_{\zeta'=0^-} \), ... may jump
→ Analogy to electrodynamics
→ Integrating the field and wave equations over a small cylinder centered on the disc and applying Gauss theorem
→ Potentials \( \beta' \) and \( b' \) are odd functions in \( \zeta' \)
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

Calculation

Wave and field equations:

with: \( u \phi' = 0 \), \( u' = e^{-U'} \), \( \mu = \sigma(\varrho) e^{2U - 2k} \delta(\zeta') \), \( f = e^{2U} \), \( h = e^{2k} \)

\[
\nabla \cdot \left[ \frac{f'}{\varrho'^2} \left( a' \nabla A_t' - \nabla A_\varphi' \right) \right] = 0 \tag{14}
\]

\[
\nabla \cdot \left[ \frac{a' f'}{\varrho'^2} \left( a' \nabla A_t' - \nabla A_\varphi' \right) - \frac{1}{f'} \nabla A_t' \right] = -4\pi \sigma e^{-U'} \delta(\zeta') \tag{15}
\]

\[
\nabla \cdot \left[ \frac{f'^2}{\varrho'^2} \nabla a' + 4 \frac{f'}{\varrho'^2} A_t' \left( a' \nabla A_t' - \nabla A_\varphi' \right) \right] = 0 \tag{16}
\]

\[
\nabla \cdot \nabla f' - \frac{1}{f'} (\nabla f')^2 + \frac{f'^3}{\varrho'^2} (\nabla a')^2 - 2 \left[ (\nabla A_t')^2 + \frac{f'^2}{\varrho'^2} (a' \nabla A_t' - \nabla A_\varphi')^2 \right] = 8\pi \sigma f' \delta(\zeta') \tag{17}
\]

\[
k'_t,\zeta' = \frac{\varrho'}{2} \left( 4U'_t,\varphi',\zeta' - \frac{f'^2}{\varrho'^2} a'_t,\varphi',\zeta' \right)
\]

\[
+ 2 \left[ \frac{f'}{\varrho'} (A_\varphi',\varphi' - a' A_t',\varphi') \left( A_\varphi',\zeta' - a' A_t',\zeta' \right) - \frac{\varrho'}{f'} A_t',\varphi' A_t',\zeta' \right] \tag{18}
\]
The boundary value problem for a rigidly rotating disc of charged dust

- The boundary conditions
- Calculation

- Wave and field equations:
  with: \( u^{\varphi'} = 0, \quad u^t = e^{-U'}, \quad \mu = \sigma (\varphi) e^{2U - 2k \delta (\zeta)}, \quad f = e^{2U}, \quad h = e^{2k} \)

\[
\nabla \cdot \left[ \frac{f'}{g'} \left( a' \nabla A_t' - \nabla A_{\varphi'} \right) \right] = 0
\]

\( (14) \)

\[
\nabla \cdot \left[ \frac{a' f'}{g'^2} \left( a' \nabla A_t' - \nabla A_{\varphi'} \right) - \frac{1}{f'} \nabla A_{t'} \right] = -4\pi \sigma e^{-U'} \delta (\zeta')
\]

\( (15) \)

\[
\nabla \cdot \left[ \frac{f'^2}{g'^2} \nabla a' + 4 \frac{f'}{g'^2} A_{t'} \left( a' \nabla A_t' - \nabla A_{\varphi'} \right) \right] = 0
\]

\( (16) \)

\[
\nabla \cdot \nabla f' - \frac{1}{f'} (\nabla f')^2 + \frac{f'^3}{g'^2} (\nabla a')^2 - 2 \left[ (\nabla A_{t'})^2 + \frac{f'^2}{g'^2} \left( a' \nabla A_{t'} - \nabla A_{\varphi'} \right)^2 \right] = 8\pi \sigma f' \delta (\zeta')
\]

\( (17) \)

\[
k'_{\varphi, t} = \frac{g'}{2} \left( 4U' a' U' - \frac{f'^2}{g'^2} a' a' \right)
\]

\[
\quad + 2 \left[ \frac{f'}{g'} (A_{\varphi', \varphi'} - a' A_{t', \varphi'}) (A_{\varphi', \zeta'} - a' A_{t', \zeta'}) - \frac{g'}{f'} A_{t', \varphi'} A_{t', \zeta'} \right]
\]

\( (18) \)

- Equation \((14) \rightarrow (A_{\varphi', \zeta'} - a' A_{t', \zeta'}) \big|_{\zeta'=0^+} = 0\)
Wave and field equations:

with: \( u^{\phi} = 0 \), \( u^t = e^{-U} \), \( \mu = \sigma (\varrho) e^{2U-2k} \delta (\zeta) \), \( f = e^2U \), \( h = e^{2k} \)

\[
\nabla \cdot \left[ \frac{f'}{\varrho'^2} (a' \nabla A^t - \nabla A^{\phi'}) \right] = 0
\]

(14)

\[
\nabla \cdot \left[ \frac{a' f'}{\varrho'^2} (a' \nabla A^t - \nabla A^{\phi'}) - \frac{1}{f'} \nabla A^t \right] = -4\pi \sigma e^{-U} \delta (\zeta')
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(15)

\[
\nabla \cdot \left[ \frac{f'^2}{\varrho'^2} \nabla a' + 4 \frac{f'}{\varrho'^2} A^t (a' \nabla A^t - \nabla A^{\phi'}) \right] = 0
\]

(16)

\[
\nabla \cdot \nabla f' - \frac{1}{f'} (\nabla f')^2 + \frac{f'^3}{\varrho'^2} (\nabla a')^2 - 2 \left[ (\nabla A^t)^2 + \frac{f'^2}{\varrho'^2} (a' \nabla A^t - \nabla A^{\phi'})^2 \right] = 8\pi \sigma f' \delta (\zeta')
\]

(17)

\[
k'_{,\zeta'} = \frac{\varrho'}{2} \left( 4U^{,\varrho'} U^{,\zeta'} - \frac{f'^2}{\varrho'^2} a'^{,\varrho'} a'^{,\zeta'} \right)
\]

\[
+ 2 \left[ \frac{f'}{\varrho'} (A_{,\varrho'}^{,\varrho'} - a'^{,\varrho'} A^t,\varrho') (A^{,\zeta'}_{,\varrho'} - a'^{,\zeta'} A^t,\zeta') - \frac{\varrho'}{f'} A^{,\varrho'}_{,\varrho'} A^t,\zeta' \right]
\]

(18)

Equation (14) \( \rightarrow (A^{,\varrho'}_{,\varrho'} - a'^{,\varrho'} A^t,\varrho') \mid_{\zeta'=0^+} = 0 \) \( \rightarrow \beta'^{,\varrho'} = 0 \) \( \rightarrow (1) : \beta' = 0 \)
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

- Wave and field equations:
  - with: \( u' \phi' = 0, \quad u' = e^{-U'}, \quad \mu = \sigma (\phi) e^{2U-2k} \delta (\zeta'), \quad f = e^{2U}, \quad h = e^{2k} \)

\[
\nabla \cdot \left[ \frac{f'}{\rho'2} (a' \nabla A_{t'} - \nabla A_{\phi'}) \right] = 0 \quad (14)
\]

\[
\nabla \cdot \left[ \frac{a' f'}{\rho'2} (a' \nabla A_{t'} - \nabla A_{\phi'}) - \frac{1}{f'} \nabla A_{t'} \right] = -4\pi \sigma \epsilon e^{-U'} \delta (\zeta') \quad (15)
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\[
\nabla \cdot \left[ \frac{f'^2}{\rho'2} \nabla a' + 4 \frac{f'}{\rho'2} A_{t'} (a' \nabla A_{t'} - \nabla A_{\phi'}) \right] = 0 \quad (16)
\]

\[
\nabla \cdot \nabla f' - \frac{1}{f'} (\nabla f')^2 + \frac{f'^3}{\rho'^2} (\nabla a')^2 - 2 \left( \nabla A_{t'} \right)^2 + \frac{f'^2}{\rho'^2} (a' \nabla A_{t'} - \nabla A_{\phi'})^2 \right] = 8\pi \sigma f' \delta (\zeta') \quad (17)
\]

\[
\kappa'_{\zeta'} = \frac{\rho'}{2} \left( 4U'_{\phi'} U'_{\zeta'} - \frac{f'^2}{\rho'^2} a'_{\phi'} a'_{\zeta'} \right) + 2 \left[ \frac{f'}{\rho'} (A_{\phi'_{\phi'}} - a' A_{t'_{\phi'}}) (A_{\phi'_{\zeta'}} - a' A_{t'_{\zeta'}}) - \frac{\rho'}{f'} A_{t'_{\phi'}} a' A_{t'_{\zeta'}} \right] \quad (18)
\]

- Equation (14) \( \rightarrow (A_{\phi'_{\phi'}} - a' A_{t'_{\phi'}}) \bigg|_{\zeta'=0^+} = 0 \quad \rightarrow \beta'_{\phi'} = 0 \quad \rightarrow (1) : \beta' = 0 \)

- Equation (15) \( \rightarrow A_{t'_{\phi'}} \bigg|_{\zeta'=0^+} = 2\pi \sigma \epsilon e^U' \)
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

Calculation

Wave and field equations:
with: \( u^t' = 0, \quad u^t' = e^{-U'}, \quad \mu = \sigma (\varrho) e^{2U - 2k} \delta (\zeta), \quad f = e^2U, \quad h = e^2k \)

\[
\nabla \cdot \left[ \frac{f'}{\varrho'2} (a' \nabla A_{t'} - \nabla A_{\varphi'}) \right] = 0
\]

\[
\nabla \cdot \left[ \frac{a'f'}{\varrho'2} (a' \nabla A_{t'} - \nabla A_{\varphi'}) - \frac{1}{f'} \nabla A_{t'} \right] = -4\pi \sigma e^{-U'} \delta (\zeta')
\]

\[
\nabla \cdot \left[ \frac{f'^2}{\varrho'2} \nabla a' + 4 \frac{f'}{\varrho'2} A_{t'} (a' \nabla A_{t'} - \nabla A_{\varphi'}) \right] = 0
\]

\[
\nabla \cdot \nabla f' - \frac{1}{f'} (\nabla f')^2 + \frac{f'^3}{\varrho'2} (\nabla a')^2 - 2 \left( \nabla A_{t'} \right)^2 + \frac{f'^2}{\varrho'2} (a' \nabla A_{t'} - \nabla A_{\varphi'})^2 \right] = 8\pi \sigma f' \delta (\zeta')
\]

\[
k'_{\varphi',\zeta'} = \frac{\varrho'}{2} \left( 4U'_{\varphi'} U'_{\zeta'} - \frac{f'^2}{\varrho'2} a'_{\varphi'} a'_{\zeta'} \right)
\]

\[
+ 2 \left[ \frac{f'}{\varrho'} (A_{\varphi',\varrho'} - a' A_{t',\varrho'}) (A_{\varphi',\zeta'} - a' A_{t',\zeta'}) - \frac{\varrho'}{f'} A_{t',\varrho'} A_{t',\zeta'} \right]
\]

Equation (14) \( \rightarrow (A_{\varphi',\zeta'} - a' A_{t',\zeta'}) \bigg|_{\zeta' = 0^+} = 0 \quad \rightarrow \beta'_{\varrho'} = 0 \quad \rightarrow (1) : \beta' = 0 \)

Equation (15) \( \rightarrow A_{t',\zeta'} \bigg|_{\zeta' = 0^+} = 2\pi \sigma \epsilon e^{U'} \)

Equation (16) \( \rightarrow a'_{\varphi',\zeta'} \bigg|_{\zeta' = 0^+} = 0 \)
The boundary value problem for a rigidly rotating disc of charged dust

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The boundary conditions

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Calculation

- Wave and field equations:
  
  With: \( u^{\varphi'} = 0, \ u^{t'} = e^{-U'}, \ \mu = \sigma (\varphi) e^{2U - 2k} \delta (\zeta'), \ f = e^{2U}, \ h = e^{2k} \)

  \[
  \nabla \cdot \left[ \frac{f'}{g'} (a' \nabla A_{t'} - \nabla A_{\varphi'}) \right] = 0
  \]
  \[
  \nabla \cdot \left[ \frac{a' f'}{g'^2} (a' \nabla A_{t'} - \nabla A_{\varphi'}) - \frac{1}{f'} \nabla A_{t'} \right] = -4\pi \sigma \epsilon e^{-U'} \delta (\zeta')
  \]
  \[
  \nabla \cdot \left[ \frac{f'^2}{g'^2} \nabla a' + 4 \frac{f'}{g'^2} A_{t'} (a' \nabla A_{t'} - \nabla A_{\varphi'}) \right] = 0
  \]
  \[
  \nabla \cdot \nabla f' - \frac{1}{f'} (\nabla f')^2 + \frac{f'^3}{g'^2} (\nabla a')^2 - 2 \left( (\nabla A_{t'})^2 + \frac{f'^2}{g'^2} (a' \nabla A_{t'} - \nabla A_{\varphi'})^2 \right) = 8\pi \sigma f' \delta (\zeta')
  \]

  \[
  k'_{,\zeta'} = \frac{g'}{2} \left( 4U'_{,\varphi'} U'_{,\zeta'} - \frac{f'^2}{g'^2} a'_{,\varphi'} a'_{,\zeta'} \right)
  + 2 \left[ \frac{f'}{g'} (A_{\varphi'}_{,\varphi'} - a' A_{t'}_{,\varphi'}) (A_{\varphi'}_{,\zeta'} - a' A_{t'}_{,\zeta'}) - \frac{g'}{f'} A_{t'}_{,\varphi'} A_{t'}_{,\zeta'} \right]
  \]

- Equation (14) \( \rightarrow (A_{\varphi'}_{,\zeta'} - a' A_{t'}_{,\zeta'}) \big|_{\zeta'=0^+} = 0 \) \( \rightarrow \beta'_{,\varphi'} = 0 \) \( \rightarrow (1) : \beta' = 0 \)
- Equation (15) \( \rightarrow A_{t'}_{,\zeta'} \big|_{\zeta'=0^+} = 2\pi \sigma \epsilon e^{U'} \)
- Equation (16) \( \rightarrow a'_{,\zeta'} \big|_{\zeta'=0^+} = 0 \) \( \rightarrow b'_{,\varphi'} = 0 \) \( \rightarrow (2) : b' = 0 \)
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

Calculation

Wave and field equations:
with: \( u^{\phi'} = 0, \ u^t = e^{-U'}, \ \mu = \sigma \rho e^{2U - 2k} \delta(\zeta'), \ f = e^{2U}, \ h = e^{2k} \)

\[
\nabla \cdot \left[ \frac{f'}{g^2} (a' \nabla A_{t'} - \nabla A_{\phi'}) \right] = 0 \tag{14}
\]

\[
\nabla \cdot \left[ \frac{a'f'}{g^2} (a' \nabla A_{t'} - \nabla A_{\phi'}) - \frac{1}{f'} \nabla A_{t'} \right] = -4\pi \sigma \epsilon e^{-U'} \delta(\zeta') \tag{15}
\]

\[
\nabla \cdot \left[ \frac{f'^2}{g'^2} \nabla a' + 4 \frac{f'}{g'^2} A_{t'} (a' \nabla A_{t'} - \nabla A_{\phi'}) \right] = 0 \tag{16}
\]

\[
\nabla \cdot \nabla f' - \frac{1}{f'} (\nabla f')^2 + \frac{f'^3}{g'^2} (\nabla a')^2 - 2 \left[ (\nabla A_{t'})^2 + \frac{f'^2}{g'^2} (a' \nabla A_{t'} - \nabla A_{\phi'})^2 \right] = 8\pi \sigma f' \delta(\zeta') \tag{17}
\]

\[
k'_{\zeta'} = \frac{g'}{2} \left( 4U'_{\phi'} U'_{\zeta'} - \frac{f'^2}{g'^2} a'_{\phi'} a'_{\zeta'} \right)
+ 2 \left[ \frac{f'}{g'} (A_{\phi'_{\zeta'} - a' A_{t'_{\zeta'}}}) (A_{\phi'_{\zeta'} - a' A_{t'_{\zeta'}}}) - \frac{g'}{f'} A_{t'_{\phi'}} A_{t'_{\zeta'}} \right] \tag{18}
\]

- Equation (14) \( \to (A_{\phi'_{\zeta'} - a' A_{t'_{\zeta'}}}) \big|_{\zeta' = 0^+} = 0 \to \beta'_{\phi'} = 0 \to (1) : \ beta' = 0 \)
- Equation (15) \( \to A_{t'_{\zeta'}} \big|_{\zeta' = 0^+} = 2\pi \sigma \epsilon e^{U'} \)
- Equation (16) \( \to a'_{\zeta'} \big|_{\zeta' = 0^+} = 0 \to b'_{\phi'} = 0 \to (2) : \ b' = 0 \)
- Equation (17) \( \to U'_{\zeta'} \big|_{\zeta' = 0^+} = 2\pi \sigma \)
The boundary value problem for a rigidly rotating disc of charged dust

Boundary conditions

Calculation

Wave and field equations:

with: \( u \phi' = 0 \), \( u' = e^{-U'} \), \( \mu = \sigma (\varrho) e^{2U-2k} \delta (\zeta) \), \( f = e^{2U} \), \( h = e^{2k} \)

\[
\nabla \cdot \left[ \frac{f'}{\varrho'^2} (a' \nabla A_{t'} - \nabla A_{\phi'}) \right] = 0
\]

(14)

\[
\nabla \cdot \left[ \frac{a' f'}{\varrho'^2} (a' \nabla A_{t'} - \nabla A_{\phi'}) - \frac{1}{f'} \nabla A_{t'} \right] = -4\pi \sigma e^{-U'} \delta (\zeta')
\]

(15)

\[
\nabla \cdot \left[ \frac{f'^2}{\varrho'^2} \nabla a' + 4 \frac{f'}{\varrho'^2} A_{t'} (a' \nabla A_{t'} - \nabla A_{\phi'}) \right] = 0
\]

(16)

\[
\nabla \cdot \nabla f' - \frac{1}{f'} (\nabla f')^2 + \frac{f'^3}{\varrho'^2} (\nabla a')^2 - 2 \left[ (\nabla A_{t'})^2 + \frac{f'^2}{\varrho'^2} (a' \nabla A_{t'} - \nabla A_{\phi'})^2 \right] = 8\pi \sigma f' \delta (\zeta')
\]

(17)

\[
k',\zeta' = \frac{\varrho'}{2} \left( 4U'_{,\varrho'} U'_{,\zeta'} - \frac{f'^2}{\varrho'^2} a'_{,\varrho'} a'_{,\zeta'} \right)
\]

\[
+ 2 \left[ \frac{f'}{\varrho'} (A_{\varphi',\varrho'} - a' A_{t',\varrho'}) (A_{\varphi',\zeta'} - a' A_{t',\zeta'}) - \frac{\varrho'}{f'} A_{t',\varrho'} A_{t',\zeta'} \right]
\]

(18)

Equation (14) \( \rightarrow (A_{\varphi',\zeta'} - a' A_{t',\zeta'}) \mid_{\zeta'=0+} = 0 \) \( \rightarrow \beta_{,\varrho'} = 0 \) \( \rightarrow (1) : \beta' = 0 \)

Equation (15) \( \rightarrow A_{t',\zeta'} \mid_{\zeta'=0+} = 2\pi \sigma \epsilon e^{U'} \)

Equation (16) \( \rightarrow a_{,\varrho'} \mid_{\zeta'=0+} = 0 \) \( \rightarrow b_{,\varrho'} = 0 \) \( \rightarrow (2) : b' = 0 \)

Equation (17) \( \rightarrow U'_{,\zeta'} \mid_{\zeta'=0+} = 2\pi \sigma \epsilon \left(e^{U'}\right)_{,\zeta'} \mid_{\zeta'=0+} = (3) : A_{t',\zeta'} \mid_{\zeta'=0+} = \epsilon \left(e^{U'}\right)_{,\zeta'} \mid_{\zeta'=0+} \)
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

Calculation

Wave and field equations:
with: \( u^{\phi'} = 0 \), \( u^t = e^{-U'} \), \( \mu = \sigma (\varrho) e^{2U-2k} \delta (\zeta) \), \( f = e^{2U} \), \( h = e^{2k} \)

\[
\nabla \cdot \left[ \frac{f'}{\varrho'^2} (a'^{'} A^t - \nabla A^{\phi'}) \right] = 0
\]
(14)

\[
\nabla \cdot \left[ \frac{a'^{'} f'}{\varrho'^2} (a'^{'} A^t - \nabla A^{\phi'}) - \frac{1}{f'} \nabla A^t \right] = -4\pi \sigma \varepsilon e^{-U'} \delta (\zeta')
\]
(15)

\[
\nabla \cdot \left[ \frac{f'^{2}}{\varrho'^{2}} \nabla a'^{'} + 4 \frac{f'}{\varrho'^{2}} A^t (a'^{'} A^t - \nabla A^{\phi'}) \right] = 0
\]
(16)

\[
\nabla \cdot \nabla f' - \frac{1}{f'} (\nabla f')^2 + \frac{f'^{3}}{\varrho'^{2}} (\nabla a')^2 - 2 \left[ (\nabla A^t)^2 + \frac{f'^{2}}{\varrho'^{2}} (a'^{'} A^t - \nabla A^{\phi'})^2 \right] = 8\pi \sigma f' \delta (\zeta')
\]
(17)

\[
k'^{'}_{\zeta'} = \frac{g'}{2} \left( 4U'^{'}_{\varrho'} U'^{'}_{\zeta'} - \frac{f'^{2}}{\varrho'^{2}} a'^{'} a'^{'}_{\varrho'} a'^{'}_{\zeta'} \right)
\]

\[
+ 2 \left[ \frac{f'}{\varrho'} (A^{\varphi'}_{\zeta'} - a^{'} A^t_{\zeta'}) (A^{\varphi'}_{\zeta'} - a^{'} A^t_{\zeta'}) - \frac{g'}{f'} A^{t}_{\varrho'} A^{t}_{\zeta'} \right]
\]
(18)

Equation (14) \( \rightarrow (A^{\varphi'}_{\zeta'} - a^{'} A^t_{\zeta'}) \bigg|_{\zeta'=0^+} = 0 \) \( \rightarrow \beta'^{'}_{\varrho'} = 0 \) \( \rightarrow (1) : \beta' = 0 \)

Equation (15) \( \rightarrow A^t_{\zeta'} \bigg|_{\zeta'=0^+} = 2\pi \sigma \varepsilon e^{U'} \)

Equation (16) \( \rightarrow a'^{'}_{\zeta'} \bigg|_{\zeta'=0^+} = 0 \) \( \rightarrow b'^{'}_{\varrho'} = 0 \) \( \rightarrow (2) : b' = 0 \)

Equation (17) \( \rightarrow U'^{'}_{\zeta'} \bigg|_{\zeta'=0^+} = 2\pi \sigma \) \( \rightarrow (3) : A^t_{\zeta'} \bigg|_{\zeta'=0^+} = \varepsilon (e^{U'})_{\zeta'} \bigg|_{\zeta'=0^+} \)

Equation \( (R = 8\pi \mu) \rightarrow k'^{'}_{\zeta'} \bigg|_{\zeta'=0^+} = 0 \)
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

**Calculation**

- Wave and field equations:
  
  with: \( u' = 0 \), \( u' = e^{-U} \), \( \mu = \sigma (\varrho) e^{2U} - 2k \delta (\zeta) \), \( f = e^{2U} \), \( h = e^{2k} \)

\[
\nabla \cdot \left[ \frac{f'}{\varrho''} (a' \nabla A_{t'} - \nabla A_{t'}) \right] = 0
\]

\[
\nabla \cdot \left[ \frac{\varrho' f'}{\varrho''} (a' \nabla A_{t'} - \nabla A_{t'}) - \frac{1}{f'} \nabla A_{t'} \right] = -4\pi \sigma \epsilon e^{-U} \delta (\zeta')
\]

\[
\nabla \cdot \frac{f'^2}{\varrho''} \nabla a' + 4 \frac{f'}{\varrho''} A_{t'} (a' \nabla A_{t'} - \nabla A_{t'}) = 0
\]

\[
\nabla \cdot \nabla f' - \frac{1}{f'} (\nabla f')^2 + \frac{f'^3}{\varrho''} (\nabla a')^2 - 2 \left[ (\nabla A_{t'})^2 + \frac{f'^2}{\varrho''} (a' \nabla A_{t'} - \nabla A_{t'})^2 \right] = 8\pi \sigma f' \delta (\zeta')
\]

\[
k'_{\zeta'} = \frac{\varrho'}{2} \left( 4U'_{\varrho'} U'_{\zeta'} - \frac{f'^2}{\varrho''} a'_{\varrho'} a'_{\zeta'} \right)
+ 2 \left[ \frac{f'}{\varrho'} (A_{t'\varrho'} - a'_{t'} \varrho') (A_{t'\zeta'} - a'_{t'} \varrho') \right]
+ \frac{\varrho' f'}{f'} (A_{t'\varrho'} - a'_{t'} \varrho') (A_{t'\zeta'} - a'_{t'} \varrho')
\]

- Equation (14) \( \rightarrow \) \( (A_{\varrho'\zeta'} - a'_{t'\varrho'}) \big| \zeta' = 0+ = 0 \) \( \rightarrow \beta'_{\varrho'} = 0 \) \( \rightarrow \) (1) : \( \beta' = 0 \)
- Equation (15) \( \rightarrow \) \( A_{t'\zeta'} \big| \zeta' = 0+ = 2\pi \sigma \epsilon e^{U} \)
- Equation (16) \( \rightarrow \) \( a'_{\zeta'} \big| \zeta' = 0+ = 0 \) \( \rightarrow \) \( b'_{\varrho'} = 0 \) \( \rightarrow \) (2) : \( b' = 0 \)
- Equation (17) \( \rightarrow \) \( U'_{\zeta'} \big| \zeta' = 0+ = 2\pi \sigma \) \( \rightarrow \) (3) : \( A_{t'\zeta'} \big| \zeta' = 0+ = \epsilon (e^{U'})_{\zeta'} \big| \zeta' = 0+ \)
- Equation \( (R = 8\pi \mu) \rightarrow k'_{\zeta'} \big| \zeta' = 0+ = 0 \)
- Equation (18) \( \rightarrow \) (4) : \( (e^{U'})_{\varrho'} \big| \zeta' = 0+ = \epsilon A_{t'\varrho'} \big| \zeta' = 0+ \)
Boundary conditions at the disc for $\zeta' = 0^\pm$: (corotating frame)

$$
\beta' = 0, \quad b' = 0, \quad \left( e^{U'} \right)_{,\varrho'} + \epsilon \alpha'_{,\varrho'} = 0, \quad \alpha'_{,\zeta'} + \epsilon \left( e^{U'} \right)_{,\zeta'} = 0
$$
Boundary conditions at the disc for $\zeta' = 0^\pm$: (corotating frame)

$\beta' = 0, \quad b' = 0, \quad \left(e^{U'}\right)_{,q'} + \epsilon \alpha'_{,q'} = 0, \quad \alpha'_{,\zeta'} + \epsilon \left(e^{U'}\right)_{,\zeta'} = 0$

Regularity conditions at spatial infinity: (nonrotating frame)

$\beta = 0, \quad b = 0, \quad \alpha = 0, \quad f = e^{2U} = 1$
# Table of Contents

1. **Motivation**

2. **Basic concepts**
   - The Einstein-Maxwell equations
   - The Model of Matter
   - Metric and Four-potential
   - Wave and Field equations
   - The Corotating frame

3. **The boundary conditions**
   - The disc case
   - Calculation
   - Physical interpretation
First b.c. $\beta' = 0$:

In a inertial frame equation (14) $\nabla \times \vec{H} = 0$. 

Integration gives $H_{t c} = 0$.

No local surface charge current in the co rotating frame.

Second b.c. $b' = 0$:

Analog case ($a$: gravitomagnetic potential)

No local surface mass current in the co rotating frame.

Third b.c.

$\epsilon (e U' + \zeta') = -\alpha' + \zeta' |_{\zeta' = 0} = 2\pi \sigma$, $A'_{t c} |_{\zeta' = 0} = 2\pi \epsilon e U' - \rightarrow$

Relation between mass and charge density.
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

Physical interpretation

First b.c. $\beta' = 0$:

In a inertial frame equation (14) $\nabla \times \vec{H} = 0$

$\rightarrow$ Integration gives $H_{tc} = 0$
First b.c. $\beta' = 0$:
In a inertial frame equation (14) $\rightarrow \nabla \times \vec{H} = 0$
$\rightarrow$ Integration gives $H_{tc} = 0$
$\rightarrow$ No local surface charge current in the corotating frame
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

First b.c. $\beta' = 0$:
In a inertial frame equation (14) $\nabla \times \vec{H} = 0$
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First b.c. $\beta' = 0$:
In a inertial frame equation (14) $\nabla \times \vec{H} = 0$
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Second b.c. $b' = 0$:
Analog case ($a$: gravitomagnetic potential)
$\rightarrow$ No local surface mass current in the corotating frame
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

Physical interpretation

First b.c. $\beta' = 0$:
In a inertial frame equation (14) $\nabla \times \vec{H} = 0$
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$\Rightarrow$ No local surface charge current in the corotating frame

Second b.c. $b' = 0$:
Analog case ($a$: gravitomagnetic potential)
$\Rightarrow$ No local surface mass current in the corotating frame

Third b.c. $\epsilon \left( e^{U'} \right)'_{\zeta'} = -\alpha'_{\zeta'}$:

$U', \zeta' \bigg|_{\zeta' = 0^+} = 2\pi \sigma, \quad A_t', \zeta' \bigg|_{\zeta' = 0^+} = 2\pi \sigma \epsilon e^{U'}$
$\Rightarrow$ Relation between mass and charge density
Fourth b.c. $\epsilon \alpha', \varrho' = -\left(eU'\right)'$, $\varrho'$:

Follows also from equations of motion
Fourth b.c. $\epsilon \alpha', g' = - \left( e^U' \right)' g'$:

Follows also from equations of motion

→ Force balance for one particle in the disc
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

Physical interpretation

Fourth b.c. \( \epsilon \alpha', \varrho' = - \left( e^{U'} \right)' \varrho' \):

Follows also from equations of motion

→ Force balance for one particle in the disc

\[ \epsilon (\alpha - \Omega A_\varphi), \varrho = - \left( e^U \sqrt{(1 + \Omega a)^2 - \Omega^2 \varrho^2 e^{-4U}} \right), \varrho \]
Fourth b.c. $\epsilon \alpha', \varrho' = - \left( e^{U'} \right)_{, \varrho'}$:

Follows also from equations of motion

→ Force balance for one particle in the disc

- $\epsilon (\alpha - \Omega A\varphi), \varrho = - \left( e^{U} \sqrt{(1 + \Omega a)^2 - \Omega^2 \varrho^2 e^{-4U}} \right)_{, \varrho}$

Newtonian limit:

$a \to 0$, $A\varphi \to 0$, $U \to U^G$, $\alpha \to U^E$, $v_{\varphi} = \Omega \varrho \ll 1$
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

Physical interpretation

- Fourth b.c. $\epsilon\alpha', \varrho' = -\left( e^{U'} \right)'_{\varrho'}$:

  Follows also from equations of motion

  $\rightarrow$ Force balance for one particle in the disc

  \[ \epsilon (\alpha - \Omega A_{\varphi}), \varrho = -\left( e^{U} \sqrt{(1 + \Omega a)^2 - \Omega^2 \varrho^2 e^{-4U}} \right), \varrho \]

  Newtonian limit:

  \[ a \rightarrow 0, A_{\varphi} \rightarrow 0, U \rightarrow U^G, \alpha \rightarrow U^E, v_{\varphi} = \Omega \varrho \ll 1 \]

\[ \epsilon \left( U^E \right), \varrho \approx -\left[ (1 + U^G) \sqrt{1 - \Omega^2 \varrho^2 (1 - 4U^G)} \right], \varrho \]

\[ \approx -\left[ (1 + U^G) \left( 1 - \frac{1}{2} \Omega^2 \varrho^2 \right) \right], \varrho \]

\[ \approx -\left[ U^G - \frac{1}{2} \Omega^2 \varrho^2 \right], \varrho \]
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

Physical interpretation

- Fourth b.c. $\epsilon \alpha', \varrho' = - \left( e^{U'} \right), \varrho'$:
  
  Follows also from equations of motion
  
  $\rightarrow$ Force balance for one particle in the disc
  
  - $\epsilon (\alpha - \Omega A_\varphi), \varrho = - \left( e^{U} \sqrt{(1 + \Omega a)^2 - \Omega^2 \varrho^2} e^{-4 U} \right), \varrho$
  
  - Newtonian limit:
    
    $a \to 0, A_\varphi \to 0, U \to U^G, \alpha \to U^E, v_\varphi = \Omega \varrho \ll 1$
    
    $\epsilon \left( U^E \right), \varrho \approx - \left[ \left( 1 + U^G \right) \sqrt{1 - \Omega^2 \varrho^2 (1 - 4U^G)} \right], \varrho$
    
    $\approx - \left[ \left( 1 + U^G \right) \left( 1 - \frac{1}{2} \Omega^2 \varrho^2 \right) \right], \varrho$
    
    $\approx - \left[ U^G - \frac{1}{2} \Omega^2 \varrho^2 \right], \varrho$

- ECD case $\Omega \to 0$
Conclusions
Conclusions

- For a rigidly rotating disc of charged dust we can formulate a boundary value problem for the Ernst equations

\[
(\Re \mathcal{E} + \Phi \overline{\Phi}) \Delta \mathcal{E} = (\nabla \mathcal{E} + 2\Phi \nabla \Phi) \cdot \nabla \mathcal{E}
\]

\[
(\Re \mathcal{E} + \Phi \overline{\Phi}) \Delta \Phi = (\nabla \mathcal{E} + 2\Phi \nabla \Phi) \cdot \nabla \Phi
\]

with \( \Phi = \alpha + i\beta \) and \( \mathcal{E} = (f - \Phi \overline{\Phi}) + i\beta \)
Conclusions

- For a rigidly rotating disc of charged dust we can formulate a boundary value problem for the Ernst equations

\[
(\Re \mathcal{E} + \Phi \Phi) \Delta \mathcal{E} = (\nabla \mathcal{E} + 2 \Phi \nabla \Phi) \cdot \nabla \mathcal{E}
\]

\[
(\Re \mathcal{E} + \Phi \Phi) \Delta \Phi = (\nabla \mathcal{E} + 2 \Phi \nabla \Phi) \cdot \nabla \Phi
\]

with \( \Phi = \alpha + i\beta \) and \( \mathcal{E} = (f - \Phi \Phi) + ib \)

- The boundary conditions are given
  - At spatial infinity in the nonrotating frame:
    \[
    \beta = 0, \quad b = 0, \quad \alpha = 0, \quad f = e^{2U} = 1
    \]
  - At the surface of the disc in the corotating frame:
    \[
    \beta' = 0, \quad b' = 0, \quad (e^{U'})_{,e'} + e\alpha', e' = 0, \quad \alpha', \zeta' + e (e^{U'})_{,\zeta'} = 0
    \]
Conclusions

- For a rigidly rotating disc of charged dust we can formulate a boundary value problem for the Ernst equations

\[(\Re E + \bar{\Phi}\Phi) \Delta E = (\nabla E + 2\bar{\Phi}\nabla\Phi) \cdot \nabla E\]

\[(\Re E + \bar{\Phi}\Phi) \Delta \Phi = (\nabla E + 2\bar{\Phi}\nabla\Phi) \cdot \nabla \Phi\]

with \(\Phi = \alpha + i\beta\) and \(E = (f - \bar{\Phi}\Phi) + ib\)

- The boundary conditions are given
  - At spatial infinity in the nonrotating frame:
    \[\beta = 0, \quad b = 0, \quad \alpha = 0, \quad f = e^{2U} = 1\]
  - At the surface of the disc in the corotating frame:
    \[\beta' = 0, \quad b' = 0, \quad (e^{U'})_{,\varrho'} + e\alpha'_{,\varrho'} = 0, \quad \alpha'_{,\zeta'} + e(e^{U'})_{,\zeta'} = 0\]

- The boundary conditions at the surface of the disc have good physical interpretation
The boundary value problem for a rigidly rotating disc of charged dust

The boundary conditions

Physical interpretation

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