Confinement in $G_2$ Gauge Theories

Björn H. Wellegehausen

Theoretisch-Physikalisches Institut
Research Training Group (1523) 'Quantum and Gravitational Fields'
FSU Jena

with Christian Wozar and Andreas Wipf

Meeting GRK 1523 'Quantum and Gravitational Fields'
Schloss Oppurg, 03.12.2010
1. Introduction

2. The gauge group $G_2$

3. Algorithmic considerations

4. Casimir scaling and string breaking in 3D

5. $G_2$ gauge Higgs model in 4D

6. Conclusions

7. Outlook
Yang Mills theory with gauge group $G$ (Gluodynamics)

$$S_{YM}[A] = \int d^4x \left( \frac{1}{4g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} \right), \quad A \in \mathfrak{g}$$

Polyakov Loop:

$$P(\vec{x}) = \text{tr} \mathcal{P}(\vec{x}), \quad \mathcal{P}(\vec{x}) = \frac{1}{N_c} \mathcal{T} \left( \exp i \int_0^{\beta T} A_0(\tau, \vec{x}) d\tau \right), \quad \beta_T = \frac{1}{T}$$

Free energy of an infinitely heavy quark sitting at $\vec{x}$:

$$e^{-\beta_T F_q} \propto \langle P(\vec{x}) \rangle$$

Gauge group $SU(3)$ corresponds to the bosonic sector of QCD
Why it is interesting to study $G_2$ gauge theories?

- In $SU(3)$ gauge theory confinement is related to the center of the gauge group
- The center of $G_2$ is trivial
- $G_2$ Yang Mills helps to clarify the relevance of center symmetry for confinement
- Possibility to distinguish between different confinement scenarios
- Similarity to QCD where center symmetry is explicitly broken by matter field
- Test of Casimir scaling hypothesis and string breaking in different representations of the gauge group
The gauge group $G_2$
Properties of the exceptional Lie-group $G_2$

Table: Centers $\mathcal{Z}$ of simple lie groups

<table>
<thead>
<tr>
<th>group</th>
<th>$A_r$</th>
<th>$B_r$</th>
<th>$C_r$</th>
<th>$D_{2r}$</th>
<th>$D_{2r+1}$</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
<th>$F_4$</th>
<th>$G_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>center $\mathcal{Z}$</td>
<td>$\mathbb{Z}_{r+1}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td>$\mathbb{Z}_4$</td>
<td>$\mathbb{Z}_3$</td>
<td>$\mathbb{Z}_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- $G_2$ is the smallest simple Lie-group which is simply connected and has a trivial center
- Two fundamental representations
  \[
  \{7\} = [1, 0], \quad \{14\} = [0, 1].
  \]
- It is a subgroup of $SO(7)$
- The gauge group $SU(3)$ of QCD is a subgroup of $G_2$ and the corresponding coset space is a sphere
  \[
  G_2/SU(3) \sim SO(7)/SO(6) \sim S^6.
  \]
The gauge group $G_2$
Representation theory and implications for confinement

**$SU(3)$ gluodynamics**

- The center of $SU(3)$ is $\mathbb{Z}_3$
- Quarks and anti-quarks transform under the $\{3\}$ and $\{\bar{3}\}$ representation which have 3-ality $(1)$ and $(−1)$
- Charges of quarks and anti-quarks can only be screened by particles with non-vanishing 3-ality

**Confinement in $SU(3)$ gluodynamics**

The Polyakov loop expectation value serves as an order parameter for the $\mathbb{Z}_3$ center symmetry and for confinement / deconfinement

Confinement $\iff$ Center symmetry

Static inter-quark potential is linearly rising up to arbitrary long distances
The gauge group $G_2$

Representation theory and implications for confinement

### Confinement in $G_2$ gluodynamics

- Quarks transform under the $\{7\}$, gluons under the $\{14\}$ representation.
- Similar as in $SU(3)$ two or three quarks can build a colour singlet:
  \[
  \{7\} \otimes \{7\} = \{1\} \oplus \cdots , \quad \{7\} \otimes \{7\} \otimes \{7\} = \{1\} \oplus \cdots
  \]
- In contrast, gluons can screen the colour charge of a single static quark:
  \[
  \{7\} \otimes \{14\} \otimes \{14\} \otimes \{14\} = \{1\} \oplus \cdots .
  \]
- The flux tube between two static quarks can break due to gluon production.
- No linear rising potential up to arbitrary long distances.

### Confinement in $G_2$ gluodynamic really means as in QCD

- Absence of free colour charges in the physical spectrum.
- Linear rising potential only at intermediate scales.

---

The gauge group $G_2$

The confinement-deconfinement phase transition

- The Polyakov loop is an approximate order parameter which changes rapidly at the phase transition and is small (but non-zero) in the confining phase
- First order confinement deconfinement phase transition

---

Algorithmic considerations

Lattice regularization

Continuum action:

\[ S_{\text{YM}}[A] = \int d^4x \left( \frac{1}{4g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} \right), \quad A \in \mathfrak{g} \]

Lattice action:

\[ S_{\text{YM}}[U] = \beta \sum \left( 1 - \frac{1}{N_c} \text{tr} \text{Re} U \Box \right), \quad U \in \mathcal{G} \]

\[ \beta = \frac{2N_c}{g^2 a^4}, \quad \text{Lattice spacing: } a, \quad \text{Continuum limit: } \beta \to \infty \]

- We want to use Monte Carlo methods to compute \( \langle O \rangle \)
- For \( SU(N) \) gauge theories: Heatbath algorithm
- We need an algorithm for (in principle) any semisimple Lie algebra (Hybrid-Monte-Carlo algorithms)
**Algorithmic considerations**

**LHMC algorithm**

*Free evolution* on a semisimple group is the Riemannian geodesic motion with respect to the Cartan-Killing metric $ds_G^2 = \kappa \text{tr} \left( dU U^{-1} \otimes dU U^{-1} \right)$

**Lagrangian:**

$$L = \frac{1}{2} \sum_{x, \mu} \text{tr} \left( i \dot{U}_{x, \mu} U_{x, \mu}^{-1} \right)^2 - S_{YM}[U]$$

**Hamiltonian:**

$$H = -\frac{1}{2} \sum_{x, \mu} \text{tr} \mathcal{P}_{x, \mu}^2 + S_{YM}[U]$$

with Lie algebra valued momenta:

$$\mathcal{P}_{x, \mu} = i \frac{\partial L}{\partial (\dot{U}_{x, \mu} U_{x, \mu}^{-1})} = -i \dot{U}_{x, \mu} U_{x, \mu}^{-1} \in \mathfrak{g}$$

The variation $\delta_{U_{x, \mu}} H = 0$ yields the following equations of motion

$$\dot{\mathcal{P}}_{x, \mu} = \sum_a \text{tr} \left( F_{x, \mu} T_a \right) T_a \quad \text{and} \quad \dot{U}_{x, \mu} = i \mathcal{P}_{x, \mu} U_{x, \mu}$$

with the *force* $F_{x, \mu}$ and trace-orthonormal basis $T_a$ of $\mathfrak{g}$
Casimir scaling and string breaking in 3D

The quark anti-quark potential

- The quark anti-quark potential in representation $\mathcal{R}$ can be parametrised as

$$V_\mathcal{R}(R) = \gamma_\mathcal{R} - \frac{\alpha_\mathcal{R}}{R} + \sigma_\mathcal{R}R$$

with string tension $\sigma_\mathcal{R}$ and Coulomb constant $\alpha_\mathcal{R}$.

- Lattice derivative defines the local string tension $\sigma_\mathcal{R}(R)$

$$\sigma_\mathcal{R}(R + \rho/2) = \frac{V_\mathcal{R}(R + \rho) - V_\mathcal{R}(R)}{\rho} = \frac{\alpha_\mathcal{R}}{R(R + \rho)} + \sigma_\mathcal{R}$$

### Casimir scaling

At intermediate scales we expect Casimir scaling $\sigma_\mathcal{R}/\sigma_\mathcal{R}' = c_\mathcal{R}/c_\mathcal{R}'$ with quadratic Casimir $c_\mathcal{R}$ of representation $\mathcal{R}$.

<table>
<thead>
<tr>
<th>$\mathcal{R}$</th>
<th>[1, 0]</th>
<th>[0, 1]</th>
<th>[2, 0]</th>
<th>[1, 1]</th>
<th>[0, 2]</th>
<th>[3, 0]</th>
<th>[4, 0]</th>
<th>[2, 1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_\mathcal{R}$</td>
<td>7</td>
<td>14</td>
<td>27</td>
<td>64</td>
<td>77</td>
<td>77</td>
<td>182</td>
<td>189</td>
</tr>
<tr>
<td>$c_\mathcal{R}$</td>
<td>12</td>
<td>24</td>
<td>28</td>
<td>42</td>
<td>60</td>
<td>48</td>
<td>72</td>
<td>64</td>
</tr>
<tr>
<td>$C_\mathcal{R}$</td>
<td>1</td>
<td>2</td>
<td>7/3</td>
<td>3.5</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>16/3</td>
</tr>
</tbody>
</table>
Physical units: Measure every quantity in units of $\mu = \sqrt{\sigma_7}$

Continuum extrapolation: $g^{-2} \sqrt{\sigma_7} = 0.381(2) \ (0.38778)$\footnote{D. Karabali, V.P. Nair and A. Yelnikov, The Hamiltonian Approach to Yang-Mills (2+1): An Expansion Scheme and Corrections to String Tension, Nucl. Phys B824 (2010) 387.}

In physical units no difference between different lattice spacings and volumes → close to the continuum limit
Casimir scaling and string breaking in 3D

Casimir scaling

Potential unscaled

- Lattice
  \[ L \times L \times T = 28^3, \quad \beta = 40 \]
- Wilson loops
- Two level algorithm
- Potential in 8 different representations

\[ = \{7\} \quad \text{red} \quad = \{14\} \quad \text{green} \quad = \{27\} \quad \text{blue} \quad = \{64\} \quad \text{magenta} \quad = \{77\} \quad \text{green} \quad = \{77\} \quad \text{gray} \quad = \{182\} \quad \text{purple} = \{189\} \]
Casimir scaling and string breaking in 3D

Casimir scaling

- Lattice
  \[ L \times L \times T = 28^3, \quad \beta = 40 \]
- Wilson loops
- Two level algorithm
- Potential in 8 different representations

Potential scaled with Casimir values \( C_R \)

\( \mu R \)

\( V_R / (\mu C_R) \)
Casimir scaling and string breaking in 3D

Casimir scaling

Casimir scaling in 3 dimensions works!
Casimir scaling and string breaking in 3D
Observing string breaking

- Three gluons can screen the color of a fundamental quark
  \[(7) \otimes (14) \otimes (14) \otimes (14) = (1) \oplus \cdots\]

- One gluon can screen the color of an adjoint quark
  \[(14) \otimes (14) = (1) \oplus \cdots\]

- Confining string can break if
  \[V_R(R^c) = E \approx 2m_{qg}\]

Mass of a quark-gluon bound state
Can be obtained from the correlation function

\[
C(T) = \left\langle \left( \bigotimes_{n=1}^{N(R)} F_{\mu\nu}(x) \right) \bigg| \mathcal{R} \left( \mathcal{U}_{xy} \right)_{ab} \left( \bigotimes_{n=1}^{N(R)} F_{\mu\nu}^\dagger(x) \right) \bigg| \mathcal{R}, a \rangle \mathcal{R}, b \right\rangle \propto \exp \left( -m_{qg} T \right)
\]
Casimir scaling and string breaking in 3D
Observing string breaking

- Lattice $L \times L \times T = 48^3$, $\beta = 30$ and Polyakov loops (three level algorithm)

String breaking in the fundamental and adjoint representation
$G_2$ gauge Higgs model in 4D
$SO(7)$ nonlinear sigma model

$\phi$: 7-component real scalar field

$$S_H[\phi] = \int d^4x \left( \frac{1}{2} (\partial_\mu \phi, \partial_\mu \phi) + V[\phi] \right), \quad V[\phi] = \lambda (\phi^2 - m^2)^2$$

Properties of the model
- invariant under global $SO(7)$
- global $SO(7)$ is spontaneously broken to $SO(6)$ (second order phase transition)
- 7 massive scalar fields $\rightarrow$ 1 massive and 6 massless scalar fields
$G_2$ gauge Higgs model in 4D
SO(7) nonlinear sigma model

Φ: 7-component real scalar field

\[ S_H[\Phi] = \frac{1}{2\kappa} \int d^4x \left( \partial_\mu \Phi, \partial_\mu \Phi \right), \quad \Phi^2 = 1 \]

Properties of the model

- invariant under global SO(7)
- global SO(7) is spontaneously broken to SO(6) (second order phase transition)
- 7 massive scalar fields \(\rightarrow\) 1 massive and 6 massless scalar fields
$S_{\text{YMH}}[A, \Phi] = \int d^4x \left( \frac{1}{4g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \kappa (D_\mu \Phi, D_\mu \Phi) \right) , \quad \Phi^2 = 1$
$SU(3)$ is a subgroup of $G_2$

\[ G_2/SU(3) \sim SO(7)/SO(6) \]

- fundamental representations decompose as

\[ \{7\} = \{3\} \oplus \{\bar{3}\} \oplus \{1\} \quad \text{and} \quad \{14\} = \{8\} \oplus \{3\} \oplus \{\bar{3}\} \]

If the Higgs field picks up a vev... QCD-like

- gluons transforming as $\{3\}$ and $\{\bar{3}\}$ become massive QCD-Quarks
- 8 gluons stay massless QCD-Gluons

In the limit $\lambda \to \infty$ and $m \to \infty$ ...

- massive gluons decouple
- $\mathbb{Z}_3$ center symmetry is restored

SU(3) gluodynamics
$G_2$ gauge Higgs model in 4D
Phase diagram - Phenomenology

$SU(3)$ Gluodynamics

$SU(3)$ Deconfinement

first order

second order

$SO(7)$ unbroken

$SO(7)$ broken

Confinement

$G_2$ Gluodynamics

$\beta = 1/g^2$

$\kappa \sim m^2$

Confinement in $G_2$ Gauge Theories 19 / 24
$G_2$ gauge Higgs model in 4D
Phase diagram - Phenomenology

\[ \kappa \sim m^2 \]

$SU(3)$ Gluodynamics

$SU(3)$ Deconfinement

SO(7) unbroken

SO(7) broken

Confinement

SO(7) sigma model

$G_2$ Deconfinement

first order

second order

first order

$\beta = 1/g^2$
Confinement in $G_2$ Gauge Theories
$G_2$ gauge Higgs model in 4D
Phase diagram - Phenomenology

$SU(3)$ Gluodynamics

$SU(3)$ Deconfinement

SO(7) sigma model

first order
second order

$\beta = 1/g^2$

$\kappa \sim m^2$

Confinement

Confinement in $G_2$ Gauge Theories
Polyakov loop \( \langle P \rangle \)

Higgs action susceptibility

\[
\chi(s_H) = V \left( \langle s_H^2 \rangle - \langle s_H \rangle^2 \right)
\]

\[
s_H = \frac{1}{V} \sum_{x\mu} \Phi_{x+\hat{\mu}} U_{x,\mu} \Phi_x
\]
**G2 gauge Higgs model in 4D**

Phase diagram - Larger lattice

---

**Lattices:** $12^3 \times 6$, $16^3 \times 6$, $20^3 \times 6$ and $24^3 \times 6$
Conclusions

- **Casimir scaling at intermediate scales** in 3 dimensions was confirmed for 8 different representations within 1 percent
- **String breaking at larger distances** was seen in both fundamental representations
- We have explored the **full phase diagram** of the $G_2$ gauge Higgs model


BW, A. Wipf, C. Wozar, Effective Polyakov Loop Dynamics for Finite Temperature $G(2)$ Gluodynamics, Phys. Rev. **D80** (2009)


BW, A. Wipf, C. Wozar, The Phases of $G(2)$ Yang-Mills-Higgs-Theory, to be published
Outlook
The gauge group $F_4$

<table>
<thead>
<tr>
<th>group</th>
<th>$A_r$</th>
<th>$B_r$</th>
<th>$C_r$</th>
<th>$D_{2r}$</th>
<th>$D_{2r+1}$</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
<th>$F_4$</th>
<th>$G_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>center $Z$</td>
<td>$\mathbb{Z}_{r+1}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td>$\mathbb{Z}_4$</td>
<td>$\mathbb{Z}_3$</td>
<td>$\mathbb{Z}_2$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

- $F_4$ is the second smallest simple Lie-group which is simply connected and has a trivial center

- Four fundamental representations

  $\{26\} = [0, 0, 0, 1], \quad \{52\} = [1, 0, 0, 0], \quad \{1274\} = [0, 1, 0, 0], \quad \{273\} = [0, 0, 1, 0]$

- It has an $SO(9)$ subgroup and it is the maximal subgroup of $E(6)$

- The gauge group $E(6)$ which could be broken to the standard model gauge group $SU(3) \times SU(2) \times U(1)$ is from a technical point of view very similar to $F4$
Outlook
First lattice results

\[ \langle P_{\text{laq}} \rangle \]

Confinement in $G_2$ Gauge Theories