Non-perturbative Aspects of Nonlinear Sigma Models

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1 Motivation, Methods and Models

2 Results
   - Discretization of Supersymmetric Theories
   - Covariant FRG Analysis of Nonlinear $O(N)$ Models and Comparison to Monte Carlo RG
   - Hamiltonian Formulation of the Functional RG
   - Renormalization of Topological Operators

3 Summary
Motivation, Methods and Models

Results

Discretization of Supersymmetric Theories
Covariant FRG Analysis of Nonlinear O(\(N\)) Models and Comparison to Monte Carlo RG
Hamiltonian Formulation of the Functional RG
Renormalization of Topological Operators

Summary
Motivation

Non-perturbative methods are important for

- QCD phase diagram
- quantization of gravity?
- topological aspects like strong CP problem
- etc.

⇒ Appropriate methods should be further investigated!

Nonlinear sigma models provide useful testing ground!
Investigated non-perturbative methods:

1. QFT on the lattice

natural regularization in the UV, $\Lambda = \frac{\pi}{a}$, and in the IR, $\lambda = \frac{\pi}{L}$

discretization of the path integral:

$$Z = \int \mathcal{D}\phi \, \mu(\phi) \, e^{-S[\phi]} \quad \Rightarrow \quad \int \prod_{x \in G} d\phi_x \, \mu(\phi_x) \, e^{-S_{disc.}[\phi_x]}$$

continuum limit: $a \to 0$ for fixed $L = aN_L$
Motivation, Methods and Models

2. Functional Renormalization Group (FRG)

$\Gamma_k[\phi]$ describes interpolation from $S[\phi]$ in the UV to effective action $\Gamma[\phi]$ in the IR

$$\Gamma[\phi] \xrightarrow{k \to 0} \Gamma_k[\phi] \xrightarrow{k \to \infty} S[\phi]$$

interpolation controlled by cutoff action

$$\Delta S_k[\phi] = \frac{1}{2} \int d^d q \frac{1}{(2\pi)^d} \phi(-q) R_k(q^2) \phi(q)$$

suppression of IR modes with $q^2 < k^2$, integration over UV modes with $q^2 > k^2$

1. $\lim_{q^2/k^2 \to 0} R_k(q^2) > 0$,  
2. $\lim_{k^2/q^2 \to 0} R_k(q^2) = 0$,  
3. $\lim_{k^2 \to \infty} R_k(q^2) \to \infty$

flow equation determines scale dependence: [C. Wetterich, Phys. Lett. B301, 90 (1993)]

$$k \partial_k \Gamma_k[\phi] = \frac{1}{2} \Tr \left\{ k \partial_k R_k \left( R_k + \Gamma_k^{(2)}[\phi] \right)^{-1} \right\}$$

convention: $k \partial_k X \equiv \partial_t X \equiv \dot{X}$
Motivation, Methods and Models

Nonlinear sigma models:

\[ S = \frac{1}{2} \zeta \int d^d x \ h_{ab}(\phi) \partial_\mu \phi^a \partial_\mu \phi^b \]

Why should one study nonlinear sigma models?

- low-energy pions
- similar properties as QCD (asymptotic freedom, confinement, dynamical generated mass, instantons, ...)
- string theory
- statistical systems (e.g. Heisenberg model)
- quantum Hall effect
- structural similarities to gravity (sigma models aren’t perturbatively renormalizable in \( d > 2 \), but asymptotic safe?)
Motivation, Methods and Models

**Nonlinear O(N) models:**

Fields $\phi$ are maps to the sphere $S^{N-1} = O(N)/O(N-1)$

Parametrization in explicitly constrained fields $n \in \mathbb{R}^N$ is often useful

$$S[n] = \frac{1}{2} \zeta \int d^d x \, \partial_\mu n \partial^\mu n , \text{ with } n^2 = 1$$

Relevance in statistical models:
$N = 1$: Ising model, $N = 2$: XY model, $N = 3$: Heisenberg model, ...
**Motivation, Methods and Models**

**CP\(^n\) models:**
fields \(\phi\) are maps to the complex projective spaces \(\text{CP}^n = \text{U}(n + 1)/\left(\text{U}(n) \times \text{U}(1)\right)\)
parametrization in explicitly constrained fields \(z \in \mathbb{C}^{n+1}\):

\[
S[z] = \frac{1}{2} \zeta \int d^d x \; D_\mu z D^\mu z, \quad \text{with } \bar{z}z = 1 \text{ and } D_\mu z^i = (\partial_\mu - \bar{z} \partial_\mu z)z^i
\]

CP\(^1\) model is equivalent to nonlinear O(3) model (Hopf map \(n_i = z^\dagger \sigma_i z\))

there are instantons in \(d = 2\), the winding number of the configuration is given as

\[
Q = \frac{i}{2\pi} \int d^2 x \; \epsilon^{\mu\nu} D_\mu z D_\nu z
\]
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3 Summary
Discretization of Supersymmetric Theories

[RF, D. Körner, A. Wipf, C. Wozar, arXiv[hep-lat]:1207.6947]

**Supersymmetry:**
symmetry between fermions and bosons

\[
\{Q^I_{\alpha}, \bar{Q}^J_{\beta}\} = 2i \delta^{IJ} \gamma^\mu_{\alpha\beta} \partial_\mu = 2 \delta^{IJ} \gamma^\mu_{\alpha\beta} P_\mu, \quad I, J = 1, \ldots, N
\]

- only non-trivial extension of Poincaré symmetry
- supersymmetry provides possible solutions for:
  - hierarchy problem of the Higgs sector, unification of electroweak and strong interaction, candidate for dark matter, ...
- problem: space-time discretization on lattice breaks supersymmetry
- restoration of the symmetry in continuum limit requires particular effort
Results

Supersymmetric $O(N)$ models:

\[
S[n, \psi] = \frac{1}{2g^2} \int d^2x \, \bar{\partial}_\mu n \partial^\mu n + i \bar{\psi} \partial \psi + \frac{1}{4} (\bar{\psi} \psi)^2 ,
\]
\[
\delta_\epsilon n = i \bar{\epsilon} \psi , \quad \delta_\epsilon \psi^\alpha = (\partial n \epsilon)^\alpha + \frac{i}{2} (\bar{\psi} \psi) n \epsilon^\alpha .
\]

$O(3) \cong \text{CP}^1$ model features additional susy due to Kähler geometry of $\text{CP}^n$ models: [B. Zumino, Phys. Lett. B 87, 203 (1979)]

\[
\delta n = i n \times \bar{\epsilon} \psi , \quad \delta \psi = -n \times \partial_\mu n \gamma^\mu \epsilon - i \bar{\epsilon} \psi \times \psi .
\]

Is it possible (like in other models) to discretize the theory such, that $S_{\text{lattice}} = QS^* \text{, where } Q \text{ is a nilpotent combination of supercharges}$?

application of this approach to $O(3) \cong \text{CP}^1$ model in

[S. Catterall and S. Ghadab, JHEP 05, 044 (2004); JHEP 10, 063 (2006)]

\[ \downarrow \text{ explicit calculation show, that their ansatz breaks the } O(3) \text{ symmetry} \]
Results

Is it possible to maintain the O(3) symmetry as well as a part of the supersymmetry on the lattice?

action of the first susy on the discretized constraint $n\psi = 0$:

$$\delta_1(n_x \psi_\alpha^x) = i\bar{\epsilon}_x \psi_\alpha^x + \sum_{y \in \Lambda} n_x D_{xy}^{\alpha\beta} n_y \epsilon^\beta + \frac{i}{2}(\bar{\psi}_x \psi_x)n_x^2 \epsilon^\alpha = \sum_{y \in \Lambda} n_x D_{xy}^{\alpha\beta} n_y \epsilon^\beta$$

action of the second susy on the discretized constraint $n\psi = 0$:

$$\delta_{II}(n_x \psi_x) = i(n_x \times \bar{\epsilon}_x) \psi_x - \sum_{y \in \Lambda} n_x (n_x \times D_{xy} n_y \epsilon) - i n_x (\bar{\epsilon}_x \psi_x \times \psi_x) = 0$$

$$\delta_{II}(n_x^2) = 2i n_x (n_x \times \bar{\epsilon}_x) = 0.$$

It is impossible to maintain simultaneously the O(3) as well as a part of the supersymmetry on the lattice!

⇒ The internal symmetries of a theory are not automatically maintained if one tries to construct a supersymmetric lattice action, but have to be treated with care!
Covariant FRG Analysis of Nonlinear $O(N)$ Models and Comparison to Monte Carlo RG

[RF, A. Wipf, O. Zanusso, arXiv[hep-th]:1207.4499]

- test model for covariant techniques of the FRG (asympt. safety in $d > 2$?)
- comparison between FRG and Monte Carlo computations

\[ \Gamma_k[\phi] = \frac{1}{2} \int d^d x \, \zeta_k \, h_{ab} \, \partial_\mu \phi^a \partial^\mu \phi^b + \alpha_k \, h_{ab} \, (\nabla_\mu \partial^\mu \phi)^a (\nabla_\nu \partial^\nu \phi)^b + L_1 (h_{ab} \partial_\mu \phi^a \partial_\nu \phi^b)^2 
+ L_2 (h_{ab} \partial_\nu \phi^a \partial_\nu \phi^b)^2, \text{ mit } \nabla_\mu v^a = \partial_\mu v^a + \Gamma^a_{cb} \partial_\mu \phi^c v^b \]

covariant background field expansion: $\Gamma[\phi] \rightarrow \Gamma[\varphi, \xi]$

regulator for the fluctuation fields $\xi$:

\[ \Delta S_k[\varphi, \xi] = \frac{1}{2} \int d^d x \, \xi^a R^k_{ab}(\varphi) \xi^b \]

cannot be written as $\Delta S_k[\phi] \Rightarrow \Gamma_k$ becomes functional of two distinct fields

$\Rightarrow$ extension of ansatz, e.g. scale factor $Z$ for $\xi$
Results

\[ \Gamma_k[\phi] = \frac{1}{2} \int d^d x \; \zeta_k \; h_{ab} \partial_\mu \phi^a \partial^\mu \phi^b + \alpha_k \; h_{ab} \left( \nabla_\mu \partial^\mu \phi \right)^a \left( \nabla_\nu \partial^\nu \phi \right)^b \]

\[ + \; L_1 (h_{ab} \partial_\mu \phi^a \partial_\nu \phi^b)^2 + L_2 (h_{ab} \partial_\nu \phi^a \partial_\nu \phi^b)^2 \]

covariant FRG provides beta functions $\beta_\zeta$, $\beta_\alpha$, $\beta_{L_1}$ and $\beta_{L_2}$

$\Rightarrow$ flow diagrams

non-trivial fixed point with only one relevant direction
Qualitative correct critical properties at the phase transition:

Problem: inclusion of $L_2$ destabilizes fixed point

[RF, A. Wipf, O. Zanusso, arXiv[hep-th]:1207.4499]
Results

Monte Carlo RG

RG step consists of:
1. block spin transformation &
2. determination of effective couplings for $\Lambda' = \Lambda/2$

$\Rightarrow$ beta functions

[RF, D. Körner, B. Wellegehausen, A. Wipf, upcoming article]

- qualitative agreement concerning the structure of the flow
- deviation in the position of the fixed point (different truncation procedure)
- fixed point even including $L_2$

$\Rightarrow$ qualitative agreement between the two non-perturbative methods
Results

Hamiltonian Formulation of the Functional RG

\[ S[\pi_\mu, \phi] = \int d^d x \, \pi_\mu \partial^\mu \phi - \mathcal{H}(\pi_\mu, \phi) \]

FRG of effective Hamiltonian action \[ \Gamma[\pi_\mu, \phi] \xrightarrow{k \to 0} \Gamma_k[\pi_\mu, \phi] \xrightarrow{k \to \infty} S[\pi_\mu, \phi] \]

One can derive flow equation like in standard FRG:

\[
i\dot{\Gamma}_k[\pi_\mu, \phi] = -\text{Tr} \left\{ \left( \frac{\delta^2 \Gamma_k}{\delta \pi \delta \phi} + R^\pi_k \partial \right)^\nu \left( \frac{\delta^2 \Gamma_k}{\delta \pi \delta \pi} \right)^{-1} \frac{\partial}{\partial \pi_{\mu}} \right. \\
\left. \left[ \frac{\delta^2 \Gamma_k}{\delta \phi \delta \phi} - \left( \frac{\delta^2 \Gamma_k}{\delta \pi \delta \phi} + R^\pi_k \partial \right)^\mu \left( \frac{\delta^2 \Gamma_k}{\delta \pi \delta \pi} \right)^{-1} \left( \frac{\delta^2 \Gamma_k}{\delta \phi \delta \pi} - R^\pi_k \partial \right) \right]^{-1} \right\}
\]

explicit computation for linear sigma model agrees with standard FRG

interesting in nonlinear models

\[ S[\pi_\mu, \phi] = \int d^d x \, \pi^\mu_a \partial_\mu \phi^a - \frac{1}{2} h^{ab}(\phi) \pi^\mu_a \pi_{b, \mu} \]
Results

\[ \Gamma_k[\pi, \phi] = \int d^d x \ \pi^\mu_a \partial_\mu \phi^a + V_k(Z), \quad \text{with } Z \equiv -\frac{1}{2} h^{ab}(\phi) \pi^\mu_a \pi_{b,\mu} \]

flow for generic function \( V_k(Z) \) can be derived:

\[ \dot{V}_k(Z) = \frac{k^d}{(4\pi)^{d/2} \Gamma[d/2+1]} \left( \frac{(N-1)k^2}{4ZV_k'^2 - k^2} \right. \\
+ \left. \sum_{n=1}^\infty \frac{(-1)^n k^{2n} 4ZV_k'^2}{(4ZV_k'^2 - k^2)^{n+1}} \left( \frac{V_k''}{V_k' + 2ZV_k''} \right)^n \frac{2^{-n} \Gamma[d/2]}{\Gamma[d/2 + n]} (2Z)^n \right) \]

for polynomial ansatz \( V(Z) = \sum_{i=1}^s g_i Z^i \):

\[ \beta g_1 = g_1 - \frac{2(N-1)}{3\pi^2} g_1^2, \quad \beta g_2 = 5g_2 - \frac{8(N-1)}{3\pi^2} g_1^4 - \frac{8(3N-2)}{9\pi^2} g_1 g_2, \quad \text{etc.} \]

- agreement with standard FRG for simplest truncation
- non-trivial fixed point for each order of the expansion

\[ g_1^* = \frac{3\pi^2}{2(N-1)}, \quad g_2^* = \frac{81\pi^6}{2(N-1)^2(3N-7)}, \quad \text{etc.} \]

- at each order only one relevant direction
- but no \( N \)-dependence of \( \nu \)
Results

Renormalization of Topological Operators

[RF, arXiv[hep-th]:1208.5948]

- strong CP problem: Why is $\theta G_{a}^{\mu\nu} \tilde{G}^{a}_{\mu\nu}$ so (unobservable) small?
- expectation: topological parameters do not run
- there are yet indications that extreme scales yield contributions

[Ansel'm, logansen, JETP 69 (1989); Shifman, Vainshtein, NPB 365 (1991); Johansen, NPB 376 (1992); Reuter, MPL A 12 (1997)]

⇒ Can this effect be confirmed in another topological model?

winding number in $\mathbb{CP}^1 \cong O(3)$ model

$$Q[\phi] = \frac{1}{2\pi} \int d^2x \epsilon^{\mu\nu} \sqrt{h} \epsilon_{ab} \partial_\mu \phi^a \partial_\nu \phi^b$$

ansatz for effective action:

$$\Gamma_k[\phi] = \frac{1}{2} \zeta_k \int d^2x h_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b + \frac{i}{2\pi} \theta_k \int d^2x \epsilon^{\mu\nu} \sqrt{h} \epsilon_{ab} \alpha \partial_\mu \phi^a \partial_\nu \phi^b$$

consideration of a generalized operator with $\theta \to \theta\alpha(x)$
Results

steps in the calculation:

- covariant background field expansion
- expansion of flow equation in powers of $\alpha$

for $k > 0$:

- computation of $\beta_\theta$ by means of off-diagonal heat kernel elements
- careful treatment of a certain limit
  $\Rightarrow$ no effect in the UV due to the asymptotic freedom

the result:

$$\theta_k = \theta_\infty \text{ for } k > 0$$
Results

Special treatment of the extreme IR

construction of a representation of the Clifford algebra

\[
\Gamma_\mu \equiv \begin{bmatrix} 0 & \Omega_{\mu}^T \\ \Omega_\mu & 0 \end{bmatrix}
\quad \text{with} \quad \Omega_1 \equiv \epsilon^{ab}, \quad \Omega_2 \equiv \delta^{ab} \quad \text{and} \quad \Gamma^* = \begin{bmatrix} 1 & 2 \\ 0 & -1 & 2 \end{bmatrix}
\]

the Dirac operators \( \mathcal{D} \equiv \Gamma_\mu \nabla^\mu, \quad D \equiv \Omega_\mu \nabla^\mu, \) and \( D^T \equiv \Omega_\mu^T \nabla^\mu \) allow for the reformulation of the flow equation as

\[
\frac{i}{2\pi} \beta_\theta \int d^2 x \ e^{\mu\nu} \sqrt{h} \epsilon_{ab} \ \alpha \ \partial_\mu \varphi^a \partial_\nu \varphi^b = - \frac{i}{2\pi} \frac{\theta_k}{\zeta_k} \int d^2 x \ \alpha(x) \ \text{tr}_4 \left\{ \langle x | \Gamma^* \mathcal{D}^2 f(- \mathcal{D}^2) | x \rangle \right\}
\]

Only zero modes contribute!

their contribution to the trace can be computed by means of a heat kernel expansion and finally yields:

\[
\theta_0 = \frac{2\pi \zeta_0 (4\pi \zeta_0 - 1)}{8\pi^2 \zeta_0^2 - 6\pi \zeta_0 + \frac{33}{35}} \ \theta_\infty
\]

Topological parameter “jumps” in the extreme IR
(result relies on \( \alpha(x) \))
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Summary

Nonlinear sigma models are useful test models for non-perturbative methods

- the discretization of the supersymmetric $\text{CP}^1 \cong \text{O}(3)$ model shows that the attempt to maintain a part of the susy on the lattice can be in conflict with internal symmetries
- a covariant analysis of the nonlinear $\text{O}(N)$ models up to fourth order in the derivative could be developed, and its results agree qualitatively with the known critical properties and with Monte Carlo computations; one operator, however, destabilizes the FRG-computation
- a Hamiltonian formulation of the FRG allows for an alternative expansion of nonlinear sigma models, which is very stable, but yields $\nu(N) = \nu(\infty)$ for all $N$
- an investigation of the $\text{CP}^1 \cong \text{O}(3)$ model yields further indications that IR effects lead to a renormalization of topological operators