Elements of QFT in Curved Space-Time

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Contents of the mini-course

- GR and its limits of applicability, Planck scale. Quantum gravity and semi-classical approach. Formulation of quantum field theory on curved background.

- Covariance and renormalizability in curved space-time. Renormalization group and conformal anomaly. Anomaly-induced effective action and Starobinsky model.

- Effective approach in curved space-time. The problem of cosmological constant and running in cosmology.

Bibliography


I.Sh., Class. Q. Grav. 25 (2008) 103001 (Topical review); 0801.0216.
Lecture 1.

GR and its limits of applicability, Planck scale.
Quantum gravity and semi-classical approach.

GR and singularities.

Dimensional approach and Planck scale.

Quantum gravity and/or string theory.

Quantum Field Theory in curved space and its importance.

Formulation of classical fields in curved space.

Quantum theory with linearized parametrization of gravity.
Classical Gravity – Newton’s Law,

\[ \vec{F}_{12} = -\frac{G M_1 M_2}{r_{12}^2} \hat{r}_{12} \quad \text{or} \quad U(r) = -G \frac{M_1 M_2}{r}. \]

Newton’s law work well from laboratory up to the galaxy scale.
For galaxies one needs, presumably, to introduce a HALO of Dark Matter, which consists from particles of unknown origin, or modify the Newton’s law - MOND,

\[ F = F(\vec{r}, \vec{v}) . \]
The real need to modify Newton gravity was because it is not relativistic while the electromagnetic theory is.

- Maxwell 1868...
- Lorentz 1895...
- Einstein 1905

**Relativity:** instead of **space + time**, there is a unique **space-time** $M_{3+1}$ (**Minkowski space**). Its coordinates are

$$x^{\mu} = (ct, x, y, z).$$

The distances (intervals) are defined as

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$
How to incorporate gravity?
The Minkowski space is flat, as the surface of a table.

GR (A. Einstein, 1915):  Gravitation = space-time metric.

- Geometry shows matter how to move.
- Matter shows space how to curve.
General Relativity and Quantum Theory

General Relativity (GR) is a complete theory of classical gravitational phenomena. It proved valid in the wide range of energies and distances.

The basis of the theory are the principles of equivalence and general covariance.
There are covariant equations for the matter (fields and particles, fluids etc) and Einstein equations for the gravitational field $g_{\mu \nu}$

$$G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = 8 \pi G T_{\mu \nu} - \Lambda g_{\mu \nu}.$$ 

We have introduced $\Lambda$, cosmological constant (CC) for completeness.

The most important solutions of GR have specific symmetries.

2) Isotropic and homogeneous metric. Universe.
Spherically-symmetric solution of Schwarzschild.

This solution corresponds to the spherical symmetry in the static mass distribution and in the classical solution. The metric may depend on the distance \( r \) and time \( t \), but not on the angles \( \varphi \) and \( \theta \).

For the sake of simplicity we suppose that there is a point-like mass in the origin of the spherical coordinate system. The solution can be written in the standard Schwarzschild form

\[
\begin{align*}
    ds^2 &= \left( 1 - \frac{r_g}{r} \right) dt^2 - \frac{dr^2}{1 - r_g/r} - r^2 d\Omega.
\end{align*}
\]

where \( r_g = 2GM \).
Performing a $1/r$ expansion we arrive at the Newton potential

$$\varphi(r) = -\frac{GM}{r} + \frac{G^2 M^2}{2r^2} + \ldots$$

Schwarzschild solution has two singularities:
At the gravitational radius $r_g = 2GM$ and at the origin $r = 0$.

The first singularity is coordinate-dependent, indicating the existence of the horizon.

Light or massive particles can not propagate from the interior of the black hole to an outside observer. The $r = r_g$ horizon looks as singularity only if it is observed from the “safe” distance.

An observer can change his coordinate system such that no singularity at $r = r_g$ will be observed.

On the contrary, $r = 0$ singularity is physical and indicates a serious problem.
Indeed, the Schwarzschild solution is valid only in the vacuum and we do not expect point-like masses to exist in the nature. The spherically symmetric solution inside the matter does not have singularity.

However, the object with horizon may be formed as a consequence of the gravitational collapse, leading to the formation of physical singularity at $r = 0$.

After all, assuming GR is valid at all scales, we arrive at the situation when the $r = 0$ singularity becomes real.

Then, the matter has infinitely high density of energy, and curvature invariants are also infinite. Our physical intuition tells that this is not a realistic situation.

Something must be modified.
Standard cosmological model

Another important solution of GR is the one for the homogeneous and isotropic metric (FLRW solution).

\[ ds^2 = dt^2 - a^2(t) \cdot \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega \right), \]

Here \( r \) is the distance from some given point in the space (for homogeneous and isotropic space-time. The choice of this point is not important). \( a(t) \) is the unique unknown function,

\( k = (0, 1, -1) \) defines the geometry of the space section \( M^3 \) of the 4-dimensional space-time manifold \( M^{3+1} \).

Consider only the case of the early universe, where the role of \( k \) and \( \Lambda \) is negligible and the radiation dominates over the matter.
Radiation-dominated epoch

is characterized by the dominating radiation with the relativistic relation between energy density and pressure \( p = \rho / 3 \) and \( T^\mu_\mu = 0 \). Taking \( k = \Lambda = 0 \), we meet the Friedmann equation

\[
\dot{a}^2 = \frac{8\pi G}{3} \frac{\rho_0 a_0^4}{a^4},
\]

Solving it, we arrive at the solution

\[
a(t) = \left( \frac{4}{3} \cdot 8\pi G \rho_0 a_0^4 \right)^{1/4} \times \sqrt{t},
\]

This expression becomes singular at \( t \to 0 \). Also, in this case the Hubble constant

\[
H = \frac{\dot{a}}{a} = \frac{1}{2t}
\]

also becomes singular, along with \( \rho_r \) and with components of the curvature tensor.

The situation is qualitatively similar to the black hole singularity.
Applicability of GR

The singularities are significant, because they emerge in the most important solutions, in the main areas of application of GR.

Extrapolating backward in time we find that the use of GR leads to a problem, while at the late Universe GR provides a consistent basis for cosmology and astrophysics. The most natural resolution of the problem of singularities is to assume that

- GR is not valid at all scales.

At the very short distances and/or when the curvature becomes very large, the gravitational phenomena must be described by some other theory, more general than the GR.

But, due to success of GR, we expect that this unknown theory coincides with GR at the large distance & weak field limit.

The most probable origin of the deviation from the GR are quantum effects.
Need for quantum field theory in curved space-time.

Let us use the dimensional arguments.

The expected scale of the quantum gravity effects is associated to the Planck units of length, time and mass. The idea of Planck units is based on the existence of the 3 fundamental constants:

\[ c = 3 \cdot 10^{10} \, \text{cm/s}, \]

\[ \hbar = 1.054 \cdot 10^{-27} \, \text{erg} \cdot \text{sec}; \]

\[ G = 6.67 \cdot 10^{-8} \, \text{cm}^3/\text{sec}^2 \, \text{g}. \]

One can use them uniquely to construct the dimensions of

- **length** \[ l_P = G^{1/2} \hbar^{1/2} c^{-3/2} \approx 1.4 \cdot 10^{-33} \, \text{cm}; \]

- **time** \[ t_P = G^{1/2} \hbar^{1/2} c^{-5/2} \approx 0.7 \cdot 10^{-43} \, \text{sec}; \]

- **mass** \[ M_P = G^{-1/2} \hbar^{1/2} c^{1/2} \approx 0.2 \cdot 10^{-5} \, \text{g} \approx 10^{19} \, \text{GeV}. \]
One can use these fundamental units in a different ways.

In particle physics people use to set $c = \hbar = 1$ and measure everything in $GeV$. Indeed, for everyday life it may not be nice.

E.g., you have to schedule the meeting “just $10^{27} \text{ GeV}^{-1}$ from now”, but “15 minutes” will be, perhaps, better appreciated.

However, in the specific area, when all quantities are (more or less) of the same order of magnitude, $GeV$ units are useful.

One can measure Newton constant $G$ in $GeV$. Then $G = 1/M_P^2$ and $t_P = l_P = 1/M_P$.

Now, why do not we take $M_P$ as a universal measure for everything? Fix $M_P = 1$, such that $G = 1$. Then everything is measured in the powers of the Planck mass $M_P$.

“20 grams of butter” $\equiv$ “$10^6$ of butter”

Warning: sometimes you risk to be misunderstood!!
Status of QFT in curved space

One may suppose that the existence of the fundamental units indicates to the presence of some fundamental physics at the Planck scale.

It may be Quantum Gravity, String Theory, ... We do not know what it really is.

So, which concepts are certain?

Quantum Field Theory and Curved space-time definitely are.

Therefore, our first step should be to consider QFT of matter fields in curved space.

Different from quantum theory of gravity, QFT of matter fields in curved space is renormalizable and free of conceptual problems.

However, deriving many of the most relevant observables is yet an unsolved problem.
Formulation of classical fields on curved background

- We impose the principles of locality and general covariance.

- Furthermore, we require the symmetries of a given theory (specially gauge invariance) in flat space-time to hold for the theory in curved space-time.

- It is also natural to forbid the introduction of new parameters with the inverse-mass dimension.

These set of conditions leads to a simplest consistent quantum theory of matter fields on the classical gravitational background.

- The form of the action of a matter field is fixed except the values of a few parameters which remain arbitrary.

- The procedure which we have described above, leads to the so-called non-minimal actions.
Along with the nonminimal scheme, there is a more simple, minimal one. According to it one has to replace

\[ \partial_\mu \rightarrow \nabla_\mu, \quad \eta_{\mu\nu} \rightarrow g_{\mu\nu}, \quad d^4x \rightarrow d^4x \sqrt{-g}. \]

Below we consider the fields with spin zero (scalar), spin 1/2 (Dirac spinor) and spin 1 (massless vector).

The actions for other possible types of fields (say, massive vectors or antisymmetric \( b_{\mu\nu} \), spin 3/2, etc), can be constructed using the same approach.
Scalar field

The minimal action for a real scalar field is

\[ S_0 = \int d^4 x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V_{\text{min}}(\varphi) \right\}, \]

where \[ V_{\text{min}}(\varphi) = -\frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!} \varphi^4 \]
is a minimal potential term.

The possible nonminimal structure is

\[ S_{\text{non-min}} = \frac{1}{2} \int d^4 x \sqrt{-g} \xi \varphi^2 R. \]

The new quantity \( \xi \) is called nonminimal parameter.

Since the non-minimal term does not have derivatives of the scalar field, it should be included into the potential term, and thus we arrive at the new definition of the classical potential.

\[ V(\varphi) = -\frac{1}{2} (m^2 + \xi R) \varphi^2 + \frac{f}{4!} \varphi^4. \]
In case of the multi-scalar theory the nonminimal term is

$$\int d^4 x \sqrt{-g} \xi_{ij} \varphi^i \varphi^j R.$$ 

Further non-minimal structures involving scalar are indeed possible, for example

$$\int R^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi.$$ 

However, these structures include constants of inverse mass dimension, therefore do not fit the principles declared above.

In fact, these terms are not necessary for the construction of consistent quantum theory.
Along with the non-minimal term, our principles admit some terms which involve only metric. These terms are conventionally called “vacuum action” and their general form is the following

$$S_{\text{vac}} = S_{\text{EH}} + S_{\text{HD}}$$

where

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left\{ R + 2\Lambda \right\}.$$

is the Einstein-Hilbert action with the CC

$S_{\text{HD}}$ includes higher derivative terms. The most useful form is

$$S_{\text{HD}} = \int d^4 x \sqrt{-g} \left\{ a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2 \right\},$$

where

$$C^2(4) = R^2_{\mu\nu\alpha\beta} - 2 R^2_{\alpha\beta} + 1/3 R^2$$

is the square of the Weyl tensor in $n=4$,

$$E = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2$$

is the integrand of the $n=4$ Gauss-Bonnet topological invariant.
In the $n = 4$ case some terms in the action

$$S_{vac} = S_{EH} + S_{HD}$$

gain very special properties.

$S_{HD}$ includes a conformal invariant $\int C^2$, topological and surface terms, $\int E$ and $\int \Box R$.

The last two terms do not contribute to the classical equations of motion for the metric.

Moreover, in the FRW case $\int C^2 = \text{const}$ and only $\int R^2$ is relevant!

However, as we shall see later on, all these terms are important, for they contribute to the dynamics at the quantum level via the conformal anomaly.

The basis $E, C^2, R^2$ is, in many respects, more useful than $R^2_{\mu \nu \alpha \beta}, R^2_{\alpha \beta}, R^2$, and that is why we are going to use it here.
For the Dirac spinor the minimal procedure leads to the expression

\[ S_{1/2} = i \int d^4x \sqrt{g} \left( \bar{\psi} \gamma^\alpha \nabla_\alpha \psi - im \bar{\psi} \psi \right), \]

where \( \gamma^\mu \) and \( \nabla_\mu \) are \( \gamma \)-matrices and covariant derivatives of the spinor in curved space-time.

Let us define both these objects.

The definition of \( \gamma^\mu \) requires the tetrad (vierbein)

\[ e^\mu_a \cdot e^\nu_b = g^{\mu\nu}, \quad e^a_\mu \cdot e^b_\mu = \eta^{ab}. \]

Now, we set \( \gamma^\mu = e^\mu_a \gamma^a \), where \( \gamma^a \) is usual (flat-space) \( \gamma \)-matrix.

The new \( \gamma \)-matrices satisfy Clifford algebra in curved space-time

\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}. \]
The covariant derivative of a Dirac spinor $\nabla_\alpha \psi$ should be consistent with the covariant derivative of tensors. We suppose

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{i}{2} w^{ab}_\mu \sigma_{ab} \psi,$$

$w^{ab}_\mu$ is usually called spinor connection and

$$\sigma_{ab} = \frac{i}{2} (\gamma a \gamma b - \gamma b \gamma a).$$

The conjugated expression is

$$\nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \frac{i}{2} \bar{\psi} w^{ab}_\mu \sigma_{ab}.$$

In order to establish the form of the spinor connection, consider the covariant derivative acting on the vector $\bar{\psi} \gamma^\alpha \psi$.

$$\nabla_\mu (\bar{\psi} \gamma^\alpha \psi) = \partial_\mu (\bar{\psi} \gamma^\alpha \psi) + \Gamma^\alpha_{\mu \lambda} \bar{\psi} \gamma^\lambda \psi,$$

The solution has the form

$$w_{\mu ab} = \frac{1}{2} \left( e_\alpha^{[b} \partial_\mu e_\alpha^{a]} + \Gamma^\alpha_{\lambda \mu} e_\alpha^{[b} e_\lambda^{a]} \right).$$
The minimal generalization for massless Abelian vector field $A_\mu$ is straightforward

$$S_1 = \frac{1}{4} \int d^4x \sqrt{g} F_{\mu\nu} F^{\mu\nu},$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$.

In the non-Abelian case we have very similar structure.

$$A_\mu \rightarrow A_\mu^a,$$

$$F_{\mu\nu} \rightarrow G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf^{abc} A_\mu^b A_\nu^c.$$

In both Abelian and non-Abelian cases the minimal action keeps the gauge symmetry. The non-minimal covariant terms for spins 1/2 and 1 have inverse mass dimension and the vacuum terms are the same as before.

Interaction with external gravity does not spoil gauge invariance of a fermion or charged scalar coupled to a gauge field. Also, the Yukawa interaction can be obtained via the minimal procedure, $\int d^4x \sqrt{-g} \varphi \bar{\psi} \psi$. 

Ilya Shapiro, Lectures on curved-space QFT, February - 2012
The quantization in curved space can be performed by means of the path integral approach.

The generating functional of the connected Green functions $W[J, g_{\mu\nu}]$ is defined as

$$e^{iW[J, g_{\mu\nu}]} = \int d\Phi \ e^{iS[\Phi, g] + iJ \cdot \Phi},$$

$d\Phi$ is the invariant measure of the functional integral and $J(x)$ are independent sources for the fields $\Phi(x)$.

- The classical action is replaced by the Effective Action (EA)

$$\Gamma[\Phi, g_{\mu\nu}] = W[J(\Phi), g_{\mu\nu}] - J(\Phi) \cdot \Phi, \quad \Phi = \frac{\delta W}{\delta J},$$

which depends on the mean fields $\Phi$ and on $g_{\mu\nu}$.

The QFT in curved space, as it is formulated above, is renormalizable and consistent.
The main difference with QFT in flat space is that in curved space $EA$ depends on the background metric, $\Gamma[\Phi, g_{\mu\nu}]$.

In terms of Feynman diagrams, one has to consider graphs with internal lines of matter fields & external lines of both matter and metric. In practice, one can consider $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.
An important observation is that all those “new” diagrams with $h_{\mu\nu}$ legs have superficial degree of divergence equal or lower that the “old” flat-space diagrams.

Consider the case of scalar field which shows why the nonminimal term is necessary.

Ilya Shapiro, Lectures on curved-space QFT, February - 2012
In general, the theory in curved space can be formulated as renormalizable. One has to follow the prescription

\[ S_t = S_{\text{min}} + S_{\text{non.min}} + S_{\text{vac}}. \]

Renormalization involves fields and parameters like couplings and masses, \( \xi \) and vacuum action parameters.

**Introduction:** Buchbinder, Odintsov & I.Sh. (1992).

**Relevant diagrams for the vacuum sector**

All possible covariant counterterms have the same structure as

\[ S_{\text{vac}} = S_{\text{EH}} + S_{\text{HD}} \]
Final observation about higher derivatives

The consistent theory can be achieved only if we include

\[ S_{HD} = \int d^4x \sqrt{-g} \left\{ a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2 \right\}, \]

\[ C^2(4) = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + 1/3 R^2 \] is the square of the Weyl tensor.

In quantum gravity such a HD term means massive ghost, the gravitational spin-two particle with negative kinetic energy. This leads to the problem with unitarity, at least at the tree level.

In the present case gravity is external and unitarity of the gravitational S-matrix does not matter.

The consistency criterium includes: physically reasonable solutions and their stability under small perturbations.

J. Fabris, Ana Pelinson, Filipe Salles, I. Sh., arXiv:1112.5202; JCAP.
Conclusions

• QFT of matter fields in curved space-time is definitely a very important object of study, because it concerns real and not well understood physics.

• QFT of matter fields in curved space-time can be always formulated as renormalizable theory if the corresponding theory in flat space-time is renormalizable.

• The action of QFT of matter fields in curved space-time includes non-minimal term in the scalar sector and additional higher derivative terms in the vacuum (gravity) sector.

• Different from QG, the higher derivative terms do not pose a problem, because we do not need physical interpretation for the gravitational propagator.