Numerically tractable formulations of the Einstein equations

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Graduiertenkolleg "Quantum and Gravitational Fields"
University of Jena
16 June 2009

Outline of the talk

1. Introduction
2. Cauchy problem
3. Stability of the constraints
4. Initial-boundary value problem
5. Conclusions

Outline of the talk

Einstein's equations in a nutshell

- Spacetime is a four-dimensional smooth Lorentzian manifold with metric g_{ab}
- Levi-Civita connection $\nabla g = 0$, Christoffel symbols

$$\Gamma^c{}_{ab} = \tfrac{1}{2} g^{cd} (g_{da,b} + g_{db,a} - g_{ab,d})$$

- Riemann curvature tensor

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}$$

- Ricci tensor $R_{bd} = R^a{}_{bad}$, Ricci scalar $R = g^{ab} R_{ab}$ and Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$
- Einstein's equations (1915) are

$$G_{ab} = \kappa T_{ab},$$

where T_{ab} is the energy-momentum tensor

Structure of Einstein's equations

- Introduce time coordinate $x^0 \equiv t$, spatial indices $i, j, \ldots = 1, 2, 3$
- Einstein's equations $G_{ab} = \kappa T_{ab}$ split into
  - $(ab = ij)$ 6 evolution equations containing 2nd time derivatives
  - $(ab = 0b)$ 4 constraint equations containing no 2nd time derivatives
- Constraints are preserved by the evolution equations: contracted Bianchi identities $\nabla_b G^{ab} = 0$ read

$$\begin{aligned} G^{0i}{}_{,0} + G^{ij}{}_{,j} &= f_1(G^{00}, G^{0i}, G^{ij}), \\ G^{00}{}_{,0} + G^{0i}{}_{,i} &= f_2(G^{00}, G^{0i}, G^{ij}), \end{aligned}$$

  so if $G^{00} = G^{0i} = 0$ initially then they vanish at all times
- Cf. Maxwell's equations in vacuo,

$$\nabla \cdot E = 0, \quad \nabla \cdot B = 0, \quad \partial_t E = \nabla \wedge B, \quad \partial_t B = -\nabla \wedge E$$

$$\Rightarrow \partial_t(\nabla \cdot E) = \nabla \cdot (\nabla \wedge B) = 0, \quad \text{similarly} \;\; \partial_t(\nabla \cdot B) = 0$$

Cauchy problem for Einstein's equations

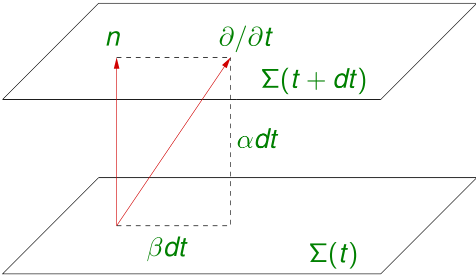
- Choose initial spacelike hypersurface $\Sigma : t = 0$
- Specify g_{ab} and $\partial_t g_{ab}$ on Σ satisfying the constraint equations
- Solve evolution equations to determine g_{ab} and $\partial_t g_{ab}$ for $t > 0$

- Cauchy problem must be well posed: for given initial data a unique solution must exist and depend continuously on the data

- There are 10 unknowns g_{ab} but only 6 evolution equations?!
- But note Einstein's equations transform covariantly under arbitrary coordinate transformations $x^{a'} = A^{a'}{}_a x^a$: $G^{a'b'} = A^{a'}{}_a A^{b'}{}_b G^{ab}$
  $\Rightarrow$ we must impose 4 coordinate or gauge conditions on g_{ab}

ADM formulation 1

(Arnowitt, Deser & Misner 1962)



- $$(\tfrac{\partial}{\partial t})^a = \beta^a + \alpha n^a,$$

  n^a unit timelike normal to Σ,
  α lapse function, β^i shift vector

- Metric takes the form

$$g = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

  where γ_{ij} is spatial metric induced on Σ

- Also define the extrinsic curvature of Σ,

$$K_{ij} = -\gamma_i{}^a \gamma_j{}^b \nabla_{(a} n_{b)} = -\tfrac{1}{2}\mathcal{L}_n \gamma_{ij}$$

ADM formulation 2

- D is covariant derivative of γ, R_{ij} and R Ricci tensor/scalar of γ, $K \equiv \gamma^{ij} K_{ij}$
- Vacuum for simplicity
- Hamiltonian and momentum constraints

$$\begin{aligned} \mathcal{H} &= R + K^2 - K_{ij} K^{ij} = 0 \\ \mathcal{M}^j &= D_i (K^{ij} - \gamma^{ij} K) = 0 \end{aligned}$$

- Evolution equations

$$\begin{aligned} \partial_0 \gamma_{ij} &= -2\alpha K_{ij}, \\ \partial_0 K_{ij} &= -D_i D_j \alpha + \alpha (R_{ij} - 2 K_{ik} K^k{}_j + K_{ij} K), \end{aligned}$$

  where $\partial_0 \equiv \partial_t - \mathcal{L}_\beta$
- Supplemented by gauge conditions determining α and β^i

Outline of the talk

Well posedness of the Cauchy problem 1

- Consider linear first-order system of PDEs

$$\begin{aligned} \partial_t \mathbf{u}(t,\mathbf{x}) &= A^k \partial_k \mathbf{u}(t,\mathbf{x}) + B\mathbf{u}(t,\mathbf{x}), \\ \mathbf{u}(0,\mathbf{x}) &= \mathbf{f}(\mathbf{x}) \end{aligned}$$

  where $\mathbf{u}$ is a vector and A^k, B are constant square matrices

- Take Fourier transform (simple wave solutions), ω_k real

$$\begin{aligned} \mathbf{u}(t,\mathbf{x}) &= e^{i\omega_k x^k} \hat{\mathbf{u}}(t,\omega) \\ \Rightarrow \partial_t \hat{\mathbf{u}} &= iA^k \omega_k \hat{\mathbf{u}} \equiv i|\omega| A^k \omega'_k \hat{\mathbf{u}} \equiv i|\omega| A(\omega')\hat{\mathbf{u}} \end{aligned}$$

Hyperbolicity of first-order systems

- weakly hyperbolic if for every ω' the eigenvalues of $A(\omega')$ are real
- strongly hyperbolic if in addition there is a complete set of linearly independent eigenvectors

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- Not weakly hyperbolic $\Rightarrow A(\omega')$ has complex eigenvalues $\Rightarrow \partial_t \hat{\mathbf{u}} = i|\omega| A(\omega')\hat{\mathbf{u}}$ has exponentially growing solutions
- Strongly hyperbolic $\Rightarrow$ well posed, energy estimate

$$\|\mathbf{u}(t,\cdot)\| \leqslant K e^{\alpha t} \|\mathbf{f}(\cdot)\|$$

where constants K, α are independent of $\mathbf{f}$ and norms are L_2

Proof.

Diagonalize $A(\omega') = S^{-1}\Lambda S$ with Λ real. Then $\hat{\mathbf{u}}' \equiv S^{-1}\hat{\mathbf{u}}$ obeys $\partial_t \hat{\mathbf{u}}' = (i|\omega|\Lambda + B')\hat{\mathbf{u}}'$ with $B' \equiv S^{-1}BS$ and hence

$$\begin{aligned}
\partial_t |\hat{\mathbf{u}}'|^2 &= \langle \partial_t \hat{\mathbf{u}}', \hat{\mathbf{u}}' \rangle + \langle \hat{\mathbf{u}}', \partial_t \hat{\mathbf{u}}' \rangle \\
&= \langle (i|\omega|\Lambda + B')\hat{\mathbf{u}}', \hat{\mathbf{u}}' \rangle + \langle \hat{\mathbf{u}}', (i|\omega|\Lambda + B')\hat{\mathbf{u}}' \rangle \\
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&\leqslant 2\alpha |\hat{\mathbf{u}}'|^2, \qquad \alpha \equiv |B'|
\end{aligned}$$

$|\hat{\mathbf{u}}'(t,\omega)| \leqslant e^{\alpha t}|\hat{\mathbf{u}}'(0,\omega)| \Rightarrow |\hat{\mathbf{u}}(t,\omega)| \leqslant K e^{\alpha t} |\hat{\mathbf{u}}(0,\omega)|, \; K \equiv |S||S^{-1}|$ □

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Well posedness of the Cauchy problem 3

- Weakly but not strongly hyperbolic ⇒ source terms $B\mathbf{u}$ can cause exponential blowup: e.g.

$$\partial_t \mathbf{u} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \partial_x \mathbf{u} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{u} \;\Rightarrow\; \partial_t \hat{\mathbf{u}} = \begin{pmatrix} i\omega & i\omega \\ 1 & i\omega \end{pmatrix} \hat{\mathbf{u}}$$

  Eigenvalues are $\lambda = i\omega \pm \sqrt{i\omega}$, i.e. $\mathrm{Re}\lambda = \pm\frac{\sqrt{2}}{2}\sqrt{|\omega|}$

- Second-order systems

$$\partial_t^2 \mathbf{u} = A^{jk} \partial_j \partial_k \mathbf{u} + B\mathbf{u}$$

Hyperbolicity of second-order systems

- weakly hyperbolic if for every ω' the eigenvalues of $A(\omega') \equiv A^{jk}\omega'_j\omega'_k$ are real *and non-negative*
- strongly hyperbolic if eigenvalues are *positive* and there is a complete set of eigenvectors

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Well posedness of the Cauchy problem 4

Further extensions:

- Systems with *variable coefficients*

$$\partial_t \mathbf{u} = A^k(t, \mathbf{x}) \partial_k \mathbf{u} + B(t, \mathbf{x}) \mathbf{u} + \mathbf{c}(t, \mathbf{x})$$

  Consider high-frequency limit, freeze coefficients to constants, again strong hyperbolicity $\Rightarrow$ well posedness

- *Quasilinear* systems

$$\partial_t \mathbf{u} = A^k(t, \mathbf{x}, \mathbf{u}) \partial_k \mathbf{u} + \mathbf{b}(t, \mathbf{x}, \mathbf{u})$$

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Application to ADM formulation

(Kreiss & Ortiz 2001; Nagy, Ortiz & Reula 2004)

- Linearizing about flat space $\gamma_{ij} = \delta_{ij} + h_{ij}$, $\alpha = 1 + a$, assuming $\beta^i = 0$, and combining the two evolution equations,

$$\partial_t^2 h_{ij} = \partial_k \partial^k h_{ij} + \partial_i \partial_j h_k{}^k - \partial_i \partial_k h_j{}^k - \partial_j \partial_k h_i{}^k + 2\partial_i \partial_j a$$

- For $a = 0$ this is only weakly hyperbolic (0 is an eigenvalue)
- Linearized momentum constraint is

$$0 = \mathcal{M}_i = \partial_t(\partial_i h_k{}^k - \partial_k h_i{}^k) \equiv \partial_t \tilde{\mathcal{M}}_i$$

- Adding $-2\partial_{(i}\tilde{\mathcal{M}}_{j)}$ to $\partial_t^2 h_{ij}$,

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- For $\alpha = \sqrt{\det \gamma}$ (densitized lapse) $\Rightarrow a = \frac{1}{2} h_k{}^k$,

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  wave equation, strongly hyperbolic

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BSSN formulation 1

(Baumgarte & Shapiro 1998, Shibata & Nakamura 1995)

- Conformal 3-metric $\tilde\gamma_{ij} = e^{-4\phi}\gamma_{ij}$ with $\det\tilde\gamma = 1$
- Tracefree part of extrinsic curvature $A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}K$, $\tilde A_{ij} \equiv e^{-4\phi}A_{ij}$
- Evolution equations become

$$\begin{aligned}
\partial_0\phi &= -\tfrac{1}{6}\alpha K, \\
\partial_0\tilde\gamma_{ij} &= -2\alpha\tilde A_{ij}, \\
\partial_0 K &= -\gamma^{ij}D_jD_i\alpha + \alpha(\tilde A_{ij}\tilde A^{ij} + \tfrac{1}{3}K^2), \\
\partial_0\tilde A_{ij} &= e^{-4\phi}(-D_iD_j\alpha + \alpha R_{ij})^{\text{tf}} + \alpha(K\tilde A_{ij} - 2\tilde A_{il}\tilde A^l{}_j)
\end{aligned}$$

- Here $R_{ij} = R^\phi_{ij} + \tilde R_{ij}$ with

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BSSN formulation 2

- where $\tilde\Gamma^i \equiv \tilde\gamma^{jk}\tilde\Gamma^i{}_{jk} = -\partial_j\tilde\gamma^{ij}$ obeys

$$\begin{aligned}\partial_t\tilde\Gamma^i &= -2\tilde A^{ij}\partial_j\alpha + 2\alpha(\tilde\Gamma^i{}_{jk}\tilde A^{kj} - \tfrac{2}{3}\tilde\gamma^{ij}\partial_j K + 6\tilde A^{ij}\partial_j\phi) \\ &\quad +\partial_j(\beta^k\tilde\partial_k\gamma^{ij} - 2\tilde\gamma^{k(j}\partial_k\beta^{i)} + \tfrac{2}{3}\tilde\gamma^{ij}\partial_k\beta^k) \\ &= 2\partial_j(\gamma^{1/3}\Sigma^{ij}),\end{aligned}$$

  where $\Sigma_{ij} = \frac{1}{2}\gamma^{1/3}\partial_t\tilde\gamma_{ij}$ "distortion tensor"
- Popular gauge choice: $1+\log$ lapse

$$\partial_0\alpha = -2\alpha K$$

  combined with $\tilde\Gamma$-driver shift

$$\partial_t^2\beta^i = \tfrac{3}{4}\partial_t\tilde\Gamma^i - \eta\partial_t\beta^i$$

- Strongly hyperbolic (Beyer & Sarbach 2004)
- Used in moving puncture BBH simulations (Goddard, RIT, Jena, AEI/LSU, PSU, . . .)

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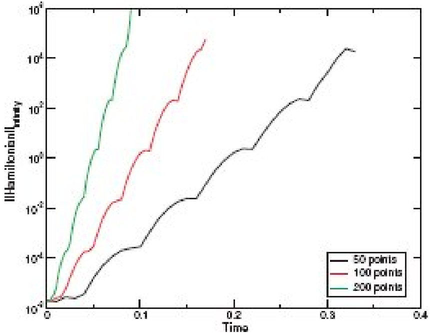
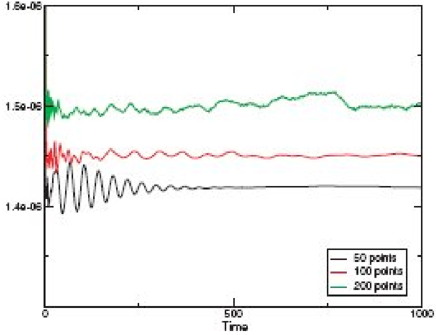
(Alcubierre *et al.* 2004 “Apples With Apples”)

Robust stability test: small random perturbation of flat space
Norm of Hamiltonian constraint for different resolutions
Left: ADM, right: BSSN





Generalized harmonic formulation

- Impose wave equation on spacetime coordinates:

$$\Box x^a = g^{bc}\nabla_b\nabla_c x^a = -g^{bc}\Gamma^a{}_{bc} = -\Gamma^a =: H^a(x),$$

where $H^a(x)$ are given functions

- Einstein's equations become

$$g^{cd}\partial_c\partial_d g_{ab} = -2\nabla_{(a}H_{b)} + 2g^{cd}g^{ef}(\partial_e g_{ca}\partial_f g_{db} - \Gamma_{ace}\Gamma_{bdf})$$

- Principal part is wave operator $\Rightarrow$ strongly hyperbolic, well posed (Fourès-Bruhat 1952)
- Used in excision BBH simulations (Pretorius, Caltech/Cornell, ...)

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Outline of the talk

The subsidiary system

- For solutions of Einstein's equations, the constraints must vanish identically
- Usually constraints are only solved initially and monitored during the evolution (free evolution)
  (alternative: constrained evolution $\rightarrow$ tomorrow's talk)
- Constraints obey closed homogeneous system of evolution equations, the subsidiary system
- Subsidiary system must be well posed
- There should not be any exponentially growing solutions (mode analysis)

Generalized harmonic formulation

- Constraints

$$\mathcal{C}_a = H_a + \Gamma_a$$

- Related to Hamiltonian and momentum constraints $\mathcal{M}_a = (\mathcal{H}, \mathcal{M}_i)$

$$\mathcal{M}_a = \tfrac{1}{2} n^b \nabla_b \mathcal{C}_a - \tfrac{1}{2}(g^{bc} n_a - n^c g^b{}_a) \nabla_b \mathcal{C}_c$$

- Main evolution equations

$$R_{ab} - \nabla_{(a}\mathcal{C}_{b)} = 0$$

- Subsidiary system

$$\nabla^b \nabla_b \mathcal{C}_a + \mathcal{C}^b \nabla_{(a}\mathcal{C}_{b)} = 0,$$

clearly strongly hyperbolic

Constraint damping

(Gundlach, Calabrese, Hinder & Martín-García 2005)

- Add terms homogeneous in constraints to evolution equations

$$R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0 \left[n_{(a}\mathcal{C}_{b)} - \tfrac{1}{2} g_{ab} n^c \mathcal{C}_c \right]$$

- Modified subsidiary system

$$\nabla^b \nabla_b \mathcal{C}_a - 2\gamma_0 \nabla^b [n_{(b}\mathcal{C}_{a)}] + \mathcal{C}^b \nabla_{(a}\mathcal{C}_{b)} - \tfrac{1}{2}\gamma_0 n_a \mathcal{C}^b \mathcal{C}_b = 0,$$

- Linearize (neglect last two terms) and make ansatz

$$\mathcal{C}_a = e^{st + i\omega_k x^k} \hat{\mathcal{C}}_a$$

- Eigenvalues are

$$s = -\tfrac{\gamma_0}{2} \pm \sqrt{(\tfrac{\gamma_0}{2})^2 - \omega^2}, \qquad s = -\gamma_0 \pm \sqrt{\gamma_0^2 - \omega^2}$$

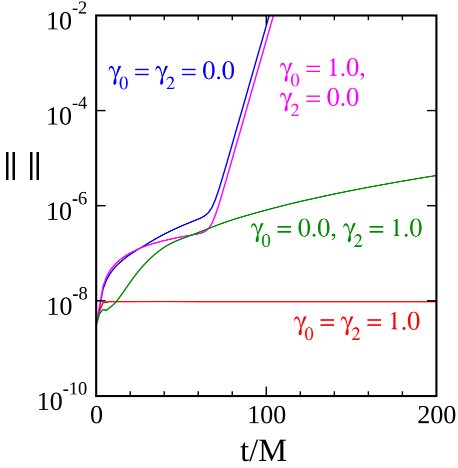
- For $\gamma_0 > 0$ all modes have $\mathrm{Re}(s) < 0$ ($\to 0$ as $\omega \to 0$)

Effect of constraint damping

(Lindblom, Scheel, Kidder, Owen & R 2006)

Norm of constraints for evolutions of Schwarzschild spacetime

γ_2 extra constraint damping parameter related to 1[st]-order reduction



ADM and BSSN

(Yoneda & Shinkai 2001/2)

ADM

- Constraints are $\mathcal{H}$ and $\mathcal{M}^i$
- Subsidiary system is strongly hyperbolic
- Eigenvalues either 0 or purely imaginary

BSSN

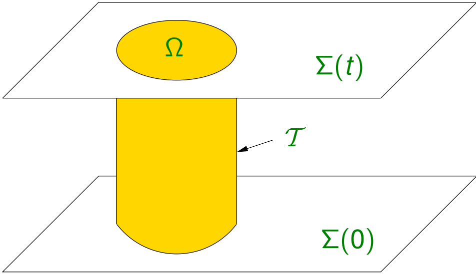
- Additional constraints $\mathcal{G}^i = \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i{}_{jk}$, $\mathcal{A} = \tilde{\gamma}^{ij}\tilde{A}_{ij}$, $\mathcal{S} = \det\tilde{\gamma} - 1$
- Subsidiary system is only weakly hyperbolic
- Eigenvalues as for ADM but more purely imaginary ones

Adding constraints to evolution equations will in general affect their hyperbolicity

Outline of the talk

Requirements for boundary conditions

- Interested in simulations of isolated systems
- Truncate domain at finite distance from the source (alternative: $\longrightarrow$ tomorrow's talk)
- Boundary conditions must
  - yield a well-posed initial-boundary value problem
  - be compatible with the constraint equations
  - be absorbing (transparent to gravitational radiation)



Well posedness of the initial-boundary value problem 1

(Kreiss & Winicour 2006)

- Half-plane problem for 2D wave equation

$$
\begin{aligned}
u_{tt} &= u_{xx} + u_{yy} + F, & 0 \leqslant x < \infty, \quad -\infty \leqslant y < \infty, \quad t \geqslant 0 \\
u(0,x,y) &= f_1(x,y), & u_t(0,x,y) = f_2(x,y), \\
u_t &= \alpha u_x + \beta u_y + q, & x = 0, \quad -\infty \leqslant y < \infty, \quad t \geqslant 0
\end{aligned}
$$

where α, β are constants with $\alpha > 0$

- Consider the homogeneous problem $F = 0, \; q = 0$.
  Suppose there is a solution of the form

$$u(t,x,y) = e^{st+i\omega y}\hat{u}(x) \qquad (*)$$

with $\mathrm{Re}(s) > 0$. Then

$$u(t,x,y) = e^{\gamma(st+i\omega y)}\hat{u}(x)$$

is a solution for any $\gamma > 0$ and hence the problem is ill posed.

Well posedness of the initial-boundary value problem 1

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Well posedness of the initial-boundary value problem 2

- Substituing $(*)$ in the wave equation,

$$\hat{u}_{xx} - (s^2 + \omega^2)\hat{u} = 0$$

General solution of ODE is

$$\hat{u} = \sigma_+ e^{\kappa_+ x} + \sigma_- e^{\kappa_- x}, \qquad \kappa_\pm = \pm\sqrt{s^2 + \omega^2}$$

Want $\sigma_+ = 0$ so that solution is bounded

- Substituting in the boundary condition,

$$(s - \alpha\kappa_- - i\beta\omega)\,\hat{u}(s, 0, \omega) = \hat{q}(s, \omega)$$

Bracket cannot vanish for $\mathrm{Re}(s) > 0$, hence we can solve for $\hat{u}$ in terms of $\hat{q}$ and estimate

$$|\hat{u}_x(s, 0, \omega)| = \sqrt{|s|^2 + \omega^2}\,|\hat{u}(s, 0, \omega)| \leqslant K|\hat{q}(s, \omega)|$$

- Inverting the Fourier-Laplace transform, obtain estimate $\forall\eta > 0$

$$\eta\|u\|^2_{\eta,1,\Omega} + \|u\|^2_{\eta,1,\mathcal{T}} \leqslant C(\eta^{-1}\|F\|^2_{\eta,0,\Omega} + \|q\|^2_{\eta,0,\mathcal{T}})$$

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Application to generalized harmonic formulation

- Linearize about flat space, frozen-coefficient approximation
- Impose $\mathcal{C}_a = 0$ at the boundary $x = 0$
- These conditions can be completed to a hierarchy of boundary conditions on components $u^{(i)}$ of the metric g_{ab}

$$
\begin{aligned}
(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2)u^{(i)} &= F^{(i)} \qquad \text{on } \Omega,\\
(\partial_t - \partial_x)u^{(i)} &= q^{(i)} \qquad \text{on } \mathcal{T},
\end{aligned}
$$

where $q^{(i)}$ depends on first derivatives of $u^{(j)}$, $j < i$, at $\mathcal{T}$

- Apply previous result for scalar wave equation step by step in the hierarchy

Higher-order absorbing boundary conditions 1

- Gravitational radiation on a flat background can be described in terms of the gauge-invariant Regge-Wheeler-Zerilli scalars $\Phi^{(\pm)}_{\ell m}$,

$$\left[\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2}\right] \Phi^{(\pm)}_{\ell m}(t,r) = 0$$

- General outgoing solution has the form

$$\Phi^{(\pm)}_{\ell m}(t,r) = \sum_{j=0}^{\ell} \frac{f^{(\pm)}_{j\ell m}(t-r)}{r^j}$$

- The following operator satisfies $B_k \Phi^{(\pm)}_{\ell m} = 0$ provided $k > \ell$:

$$B_k \equiv r^{-(2k+1)} \left[r^2(\partial_t + \partial_r)\right]^k$$

- Use this as a boundary condition (Bayliss & Turkel 1980; Buchman & Sarbach 2006/7)

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Higher-order absorbing boundary conditions 2

- These conditions can be expressed as conditions on certain components of the metric, in frozen-coefficient approximation:

$$(\partial_t - \partial_x)^m u^{(i)} = q^{(i)} \quad \text{on } \mathcal{T},$$

  where now $q^{(i)}$ depends on mth derivatives of $u^{(j)}$, $j < i$, at $\mathcal{T}$

- Well posedness result carries over (Ruiz, R & Sarbach 2007)

- Numerical implementation using pseudo-spectral methods, tested for exact solutions of the linearized Einstein equations (R, Buchman, Scheel & Pfeiffer 2009)

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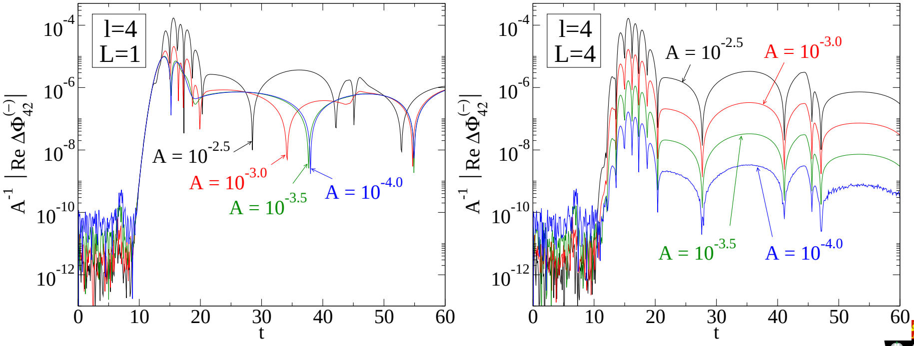
Higher-order absorbing boundary conditions 3

Outgoing $\ell = 4$ wave centered initially at $r_0 = 15$, boundary at $R = 30$
Shown is deviation of RWZ scalar Φ from exact *linearized* solution,
normalized by wave amplitude A
Left: absorbing BC of order $L = 1$, right: $L = 4$



Outline of the talk

Conclusions

- Numerical tractable formulations of the Einstein equations should
  - admit a well-posed Cauchy problem ($\rightarrow$ strong hyperbolicity)
  - avoid exponential growth of constraints ($\rightarrow$ mode analysis)
  - admit a well-posed initial boundary value problem with constraint-preserving *and absorbing* boundary conditions
- For the generalized harmonic formulation, all of these requirements have been fulfilled
- For the BSSN formulation,
  - Cauchy problem well posed
  - stability of the constraints somewhat unclear (although good performance in numerical simulations)
  - issue of suitable boundary conditions still very much open