Poincaré gauge theory of gravity:
Friedman cosmologies
with even and odd parity modes

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Local coord. \((x^1, x^2, x^3)\) at a point \(P\) of a 3D manifold and the basis vectors \((e_1, e_2, e_3)\). The basis 1-forms \(\vartheta^a = dx^a, \; a = 1, 2, 3\), are supposed to be also at \(P\). Note that \(\vartheta^1(e_1) = 1, \; \vartheta^1(e_2) = 0\), etc., i.e., \(\vartheta^a\) is dual to \(e_b: \vartheta^a(e_b) = \delta^a_b\).

Perform linear comb. of the \(\vartheta\) in order to find an arbitrary frame. In 4D, \(\vartheta^\alpha = \vartheta_i^\alpha \, dx^i\) (these will be the 4 translation potentials).
0.2 Connection

If a linear (or affine) connection is given, the parallel transfer of a vector \( C = C^\alpha e_\alpha \), can be defined, e.g.:

\[
\delta \parallel C^\beta = -\Gamma^\beta_\alpha C^\alpha,
\]

with \( \Gamma^\beta_\alpha = \Gamma_{i\alpha}^\beta dx^i \).

\( \Gamma^\beta_\alpha \) represents \( 4 \times 4 \) potentials of the 4D group of general linear transformations \( GL(4, R) \) (very similar to the Yang-Mills potential of the \( SU(3) \), say). Field strength is called curvature \( R \sim \text{curl} \, \Gamma \) or \( 16 \times 6 \) indep. comp.

\[
R_{ij\alpha} \beta \sim \partial_i \Gamma_{j\alpha} \beta - \partial_j \Gamma_{i\alpha} \beta + \text{nonl. term}, \quad R_{ij\alpha} \beta = -R_{ji\alpha} \beta.
\]

Torsion \( T^\alpha = T_{ij}^\alpha dx^i \wedge dx^j / 2 \) and curvature \( R^\alpha \beta = R_{ij\alpha} \beta dx^i \wedge dx^j / 2 \) as field strengths. Symbolically,

\[
T^\alpha = d \vartheta^\alpha + \Gamma^\alpha_\beta \vartheta^\beta, \quad R^\alpha \beta = d \Gamma^\alpha_\beta - \Gamma^\gamma_\alpha \Gamma^\beta_\gamma.
\]

Field strengths of the gauge theory of the affine group

\[
A(4, R) = R^4 \otimes GL(4, R)
\]
0.3 Metric $g(x)$

Experience points to more structure. Time and space intervals and angles should be measurable ⇒ pseudo-Riemannian (or Lorentzian) metric $g_{ij}(x) = g_{ji}(x)$. 10 indep. comp. If $g_{\alpha\beta}$ are components w.r.t. coframe, then $g_{ij} = \vartheta_i^{\alpha} \vartheta_j^{\beta} g_{\alpha\beta}$ and $g = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta$. In the 4D spacetime of SR and GR w.r.t. an orthonormal basis:

$$g_{\alpha\beta} = o_{\alpha\beta} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$ 

In the 3D space an orthonormal basis reads:

$$g_{\alpha\beta} = o_{\alpha\beta} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

We now assume that the connection of the space(time) we investigate is metric compatible. Then the nonmetricity 1-form $Q_{\alpha\beta} := -Dg_{\alpha\beta} = Q_{i\alpha\beta}dx^i$ vanishes:

$$Q_{\alpha\beta} = 0 \quad \text{or} \quad Q_{i\alpha\beta} = 0.$$ 

These are 40 constraints on the linear connection, that is, $\Gamma^\alpha_{\beta\gamma}$ depends only on 24 independent components. Spaces obeying the constraint $Q_{\alpha\beta} = 0$ are called Riemann-Cartan spaces. In this case, the length is an integrable concept. In orthonormal frames, we have then $\Gamma^\alpha_{\beta\gamma} = -\Gamma^\beta_{\alpha\gamma}$. 
0.4 Torsion and closure failure

Figure: On the geometrical interpretation of torsion, see (Erice 1995) [the corr. figure in MTW is incorrect]: The vectors $u$ and $v$ are given. At a point $P$, we transport parallelly $u$ and $v$ along $v$ or $u$. They become $u_R$ and $v_Q$. If a torsion is present, they don’t close. This is a schematic view. Note that the points $R$ and $Q$ are infinitesimally near to $P$. A proof can be found in Schouten, Ricci Calculus (1954) p.127: $db^k = \frac{1}{2} T_{ij}^k dA^{ij}$ with $dA^{ij}$ as area element and $db^k$ the closure failure, a small translation vector. By geometry, torsion is always related to a small translation! (And by physics, translation, via Noether’s theorem, is always related to energy-momentum).
0.5 Application: Cartan’s spiral staircase

É. Cartan’s Construction (1922)

The new red connection determines $F$.

Figure: A 3-dimensional space with homogeneous and isotropic torsion: the spiral staircase [see Found. Phys. 40, 1298 (2010)]
**Figure:** *Cartan’s spiral staircase* [see PRD 67,124016 (2003)]. Cartan’s rules (1922) for the construction of a homogeneous and isotropic torsion in a 3D Euclidean space: (i) A vector which is parallelly transported along itself does not change. (ii) A vector that is orthogonal to the direction of transport rotates with a prescribed constant ‘velocity’. The winding sense around the three coordinate axes is always positive. Components of the Cartan connection:

\[
\begin{align*}
\Gamma^{\alpha\beta} &= -\Gamma^{\beta\alpha} = \Gamma_{i}^{\alpha\beta} dx^{i} = \Gamma_{1}^{\alpha\beta} dx^{1} + \Gamma_{2}^{\alpha\beta} dx^{2} + \Gamma_{3}^{\alpha\beta} dx^{3}, \\
\Gamma_{1}^{23} &= \Gamma_{2}^{31} = \Gamma_{3}^{12} = \omega.
\end{align*}
\]
In exterior calculus, Cartan’s spiral staircase, on the basis of the connection components read off from our image on the last slide, turns out to be

$$\vartheta^\alpha = \delta_i^\alpha \, dx^i, \quad \Gamma^\alpha \beta = \omega^*(\vartheta^\alpha \wedge \vartheta^\beta).$$

The components of the connection are totally antisymmetric:

$$\Gamma_{\gamma \alpha \beta} = e_\gamma \lfloor \Gamma_{\alpha \beta} = \omega^*(\vartheta_\gamma \wedge \vartheta_\alpha \wedge \vartheta_\beta).$$

The Riemannian curvature vanish, $$\tilde{R}^{\alpha \beta} = 0.$$ Accordingly,

$$T^\alpha = 2 \omega^* \vartheta^\alpha, \quad \tilde{R}^{\alpha \beta} = 0, \quad R^{\alpha \beta} = \omega^2 \vartheta^\alpha \wedge \vartheta^\beta.$$
0.6 Application: Torsion in dislocation theory

**Edge dislocation** after Kröner: Disloc. line parallel to \( t \), the Burgers vector \( d_b \) perpendicular to \( t \). The gliding is characterized by \( d_g \). **Dislocation density** defined: \( d_b^k = \alpha_{ij}^k dA^{ij} \), here \( d_b^3 = \alpha_{31}^3 dA^{31} \), with \( \alpha_{31}^3 = -\alpha_{13}^3 \), hence \( \alpha_{23}^2 \neq 0 \) with \( \alpha_{\ell k} := \frac{1}{2} \epsilon_{\ell ij} \alpha_{ij}^k \).

**Screw dislocation** after Kröner: Burgers vector is parallel to \( t \). Here \( d_b^2 = \alpha_{31}^2 dA^{31} \), with \( \alpha_{31}^2 = -\alpha_{13}^2 \), hence \( \alpha_{22}^2 \neq 0 \).
Torsion in dislocation theory (continued)

Figure: Ideal cubic crystal undeformed, with ‘small’ parallelogram

Figure: Homogeneously strained crystal, parallelogram closed.

Figure: Cubic crystal deformed by edge dislocations $\alpha_{12}$. Relative orientations of the 23-lattice planes in 2-direction change:

$$d\omega_{ij} = -d\omega_{ji} = K_{kij}dx^k,$$

here for the contortion only $K_{112} = -K_{121} \neq 0$ or $K_{13} \neq 0$, with $K_{k}^{\ell} = \frac{1}{2} \epsilon^{\ell ij} K_{kij}$. We find: $K_{i}^{j} = \alpha_{i}^{j} - \frac{1}{2} \alpha_{k}^{i} \delta_{i}^{j}$ (relations to microdistorsions in a Cosserat medium, Kluge, Jena 1969).
“...the essential achievement of general relativity, namely to overcome ‘rigid’ space (ie the inertial frame), is only indirectly connected with the introduction of a Riemannian metric. The directly relevant conceptual element is the ‘displacement field’ ($\Gamma^l_{ik}$), which expresses the infinitesimal displacement of vectors. It is this which replaces the parallelism of spatially arbitrarily separated vectors fixed by the inertial frame (ie the equality of corresponding components) by an infinitesimal operation. This makes it possible to construct tensors by differentiation and hence to dispense with the introduction of ‘rigid’ space (the inertial frame). In the face of this, it seems to be of secondary importance in some sense that some particular $\Gamma$ field can be deduced from a Riemannian metric...”

A. Einstein (1955 April 04)

**Conclusion:** The inertial frames are ‘transported’ by the linear connection (the ‘displacement’ field). This links these considerations directly to the equivalence principle. In particular, the symmetry of a connection is a bastard symmetry, since it is related to two qualitatively different indices (as can be seen from the connection 1-form).
Das deutsche Original:

“...die wesentliche Leistung der allgemeinen Relativitätstheorie, nämlich die Überwindung (des) ‘starren’ Raumes, d.h. des Inertialsystems, ist nur indirekt mit der Einführung einer Riemann-Metrik verbunden. Das unmittelbar wesentliche begriffliche Element ist das die infinitesimale Verschiebung von Vektoren ausdrückende ‘Verschiebungsfeld’ ($\Gamma^l_{ik}$). Dieses nämlich ersetzt den durch das Inertialsystem gesetzten Parallelismus räumlich beliebig getrennter Vektoren (nämlich Gleichheit entsprechender Komponenten) durch eine infinitesimale Operation. Dadurch wird die Bildung von Tensoren durch Differentiation ermöglicht und so die Einführung des ‘starren’ Raumes (Inertialsystem) entbehrlich gemacht. Dem gegenüber erscheint es in gewissem Sinne von sekundärer Wichtigkeit, dass ein besonderes $\Gamma$-Feld sich aus der Existenz einer Riemann-Metrik deduzieren lässt...”

A. Einstein (1955 April 04)

A spacetime with a metric and a metric compatible connection (nonmetricity $= 0$, that is, $Q_{\alpha\beta} := -Dg_{\alpha\beta} = 0$) is called a Riemann-Cartan space $U_4$. It can either become a Weitzenböck space $W_4$, if its curvature vanishes, or a Riemann space $V_4$, if the torsion happens to vanish. These different models of spacetime are the arenas for different gravitational theories.
1. General structure of the Poincaré gauge theory (PG)

The ‘gravitational’ potentials are

\[ \vartheta^\alpha \] orthonormal coframe (weak gravity)

\[ \Gamma^{\alpha \beta} = -\Gamma^{\beta \alpha} \] Lorentz connection (strong YM-gravity)

By differentiation, we find the field strengths

\[ T^\alpha = D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta^\alpha \wedge \vartheta^\beta \] torsion

\[ R^{\alpha \beta} = d\Gamma^{\alpha \beta} - \Gamma^{\alpha \gamma} \wedge \Gamma_\gamma^\beta = -R^{\beta \alpha} \] curvature

The material currents coupled to the potentials \((\vartheta^\alpha, \Gamma^{\alpha \beta})\) are energy-momentum and spin angular momentum \((\Sigma_\alpha, \tau_{\alpha \beta})\).

The additional sources of gravity as compared to GR is the spin current \(\tau_{\alpha \beta} = -\tau_{\beta \alpha}\).

The 2 potentials span the geometry of spacetime: It is the Riemann-Cartan spacetime \(U_4\). The corresponding first order Lagrangian gauge field theory is called PG. It is a framework for gravitational gauge field theories.

Lagrangian:

\[ L_{\text{total}} = V(g_{\alpha \beta}, \vartheta^\alpha, T^\alpha, R^{\alpha \beta}) + L_{\text{matter}}(g_{\alpha \beta}, \vartheta^\alpha, \Psi, D \Psi). \]

Define the excitations (field momenta)

\[ H_\alpha = -\frac{\partial V}{\partial T^\alpha}, \quad H_{\alpha \beta} = -\frac{\partial V}{\partial R^{\alpha \beta}}, \]
2. Field equations

\[ DH_\alpha - E_\alpha = \Sigma_\alpha \quad (\delta/\delta \vartheta^\alpha: \text{1st field equation of gravity}), \]
\[ DH_{\alpha\beta} - E_{\alpha\beta} = \tau_{\alpha\beta} \quad (\delta/\delta \Gamma^{\alpha\beta}: \text{2nd field equation of gravity}), \]
\[ \frac{\delta L}{\delta \Psi} = 0 \quad (\delta/\delta \Psi: \text{matter field equation}) \]

(Einstein sector). Here energy-momentum and spin of the gauge fields are

\[ E_\alpha := e_\alpha [V + (e_\alpha] T^\beta) \wedge H_\beta + (e_\alpha] R^{\beta\gamma}) \wedge H_{\beta\gamma}, \]
\[ E_{\alpha\beta} := -\vartheta_{[\alpha} \wedge H_{\beta]} . \]

3. Einstein-Cartan theory (EC)

Simplest Lagrangian

\[ V_{EC} \sim \frac{1}{\kappa} \vartheta_i^{\alpha} \vartheta_j^{\beta} R^{ij}_{\alpha\beta}(\Gamma_k^{\gamma\delta}) \sim \frac{1}{\kappa} R \]

Einstein-Cartan (EC) theory: GR plus an add. spin contact interaction,

\[ \text{Ric} - \frac{1}{2} tr(\text{Ric}) \sim \kappa \times \Sigma \sim \kappa \times \text{energy-momentum}, \]
\[ \text{Tor} + 2 tr(\text{Tor}) \sim \kappa \times \tau \sim \kappa \times \text{spin angular momentum}. \]

Here \( \text{Ric}_{ij} := R_{kij}^{\;\;\;k}, \) \( R := \text{Ric}_i^i, \) and \( \kappa \) is Einstein’s gravitational constant \( 8\pi G/c^4. \) If spin \( \tau \to 0, \) then EC-theory \( \to \) GR, and RC-spacetime \( \to \) Riemannian spacetime. Thus, GR is included.
With $\tau \neq 0$, modified source of Einstein’s equation: $\rho \rightarrow \rho + \kappa \tau^2$ ⇒ at sufficiently high densities $\kappa \tau^2 \sim \rho$ ⇒

$$\rho_{\text{crit}} \sim \frac{m}{\lambda_{\text{Compton}}^2 \ell_{\text{Planck}}^2}$$

(result of spin-spin contact interaction),

more than $10^{52} \text{g/cm}^3$ or $10^{24} \text{K}$ for electrons, see H., v.d.Heyde, Kerlick, Nester, RMP 1976. This is valid up to $10^{-34} \text{s}$ after the big bang ($10^{-43} \text{s}$ corr. to Planck era). Spin cosmology (Kopczyński 1972, Bauerle-Haneveld 1983, Canale-de Ritis-Tarantino 1984...), spin-driven inflation (Obukhov 1993)? For parallel Dirac spins, the contact interaction is repulsive (O’Connell). The EC-theory is a viable gravitational theory, see also the books of Blagojević ‘Gravitation and Gauge Symm.’ (2002) and Ortín ‘Gravity and Strings’ (2004).

Contact interactions in particle physics were searched for by Ellerbrock, Ph.D. thesis DESY 2004, see review by Goy (2004) on HERA, LEP, Tevatron. Nothing found so far. But for EC-theory these experiments are not sensitive enough.


Their 3rd hypothesis was a ‘Torsion quintessence.’ More recently, N. Popławski, Cosmological constant from quarks and torsion, Ann. Phys. (Berlin) 2011, in press [arXiv:1005.0893]. J. D. Bjorken, arXiv:1008.0033: “We discuss vacuum condensates associated with emergent QED and with torsion, as well as...the Kodama wave function in quantum cosmology.”
4. Teleparallel equivalent GR$_{\parallel}$ of GR (see Schweizer, Straumann, and Wipf 1980)

Belongs to the class of translational gauge theories, see Hehl, Erice 1979:

\[ V_{\parallel} = \frac{1}{\kappa} V_{T^2} + R_{\alpha \beta} \wedge \lambda^\alpha \beta \quad (\lambda^\alpha \beta = \text{Lagrange multiplier}), \]

\[ V_{T^2} := -\frac{1}{2} T^\alpha \wedge ^\ast \left( - (1) T_\alpha + 2 (2) T_\alpha + \frac{1}{2} (3) T_\alpha \right). \]

Viable set! Yields local Lorentz invariance $\Rightarrow$ Einstein’s GR.

GR$_{\parallel}$ in gauge $\Gamma = 0$, Weitzenböck spacetime, field equation is Maxwell like

\[(\Omega^{k\iota}_{\iota} \sim \partial^{[k} \vartheta^{\iota]}_{\iota} + \cdots):\]

\[ D_k \Omega^{k\iota}_{\iota} + \text{nonlin. terms} \sim \kappa \times \Sigma_{\iota} \]

\[ \Box \vartheta^i_{\iota} + \text{nonlin. terms} \sim \kappa \times \Sigma_{\iota} \quad \text{(in Hilbert gauge)} \]

Compare Einstein’s equation ($g_{ij} = g_{ji}$):

\[ \Box g_{ij} + \text{nonlin. terms} \sim \kappa \times \sigma_{ij} \quad \text{(in Hilbert gauge)} \]

For scalar and for Maxwell matter, that is, for $\Sigma_{ij} = \sigma_{ij}$, it can be shown that GR$_{\parallel}$ and GR are equivalent.
5. Gravitational gauge Lagrangian with even and odd parity terms

\[
V_{PG} \sim \frac{1}{\kappa} \left( a_0 R + \Lambda + \sum_{I=1}^{3} a_{(I)} T^\alpha \wedge \star(I) T_\alpha \right) + \frac{1}{\varrho} \sum_{I=1}^{6} r_{(I)} R^{\alpha\beta} \wedge \star(I) R_{\alpha\beta} \\
+ \frac{1}{\kappa} \left( b_0 X + \sum_{1,1}^{2,3} \sigma_{(I,K)}^{(I)} T^\alpha \wedge (K) T_\alpha \right) + \frac{1}{\varrho} \sum_{1,1;2,4}^{3,6;5,5} \mu_{(I,K)}^{(I)} R^{\alpha\beta} \wedge (K) R_{\alpha\beta}.
\]

Here \( X \sim \epsilon^{ijkl} R_{[i,jkl]} \sim (3)R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta \). In Riemannian space, the whole 2nd line (the ‘shadow’ of the 1st line), with exception of \( \mu(1, 1) \), vanishes. Such type of Lagrangians have been discussed first by Obukhov, Ponomarev, Zhytnikov (1989).

\begin{itemize}
  \item \( T \rightarrow \) tensor \quad \( (2)T \rightarrow \) vector, \( \sim \mathcal{V} \), \( (3)T \rightarrow \) axial vector \( \sim \mathcal{A} \)
  \item \( R \rightarrow 10 \) Weyl, \( (2)R \rightarrow 9 \) Paircom, \( (3)R \rightarrow 1 \) Pscalar \( X \), \( (4)R \rightarrow 9 \) Ricsymf, \( (5)R \rightarrow 6 \) Riccanti, \( (6)R \rightarrow 1 \) Scalar \( R \).
\end{itemize}

With strong gravity pieces \( \sim (R^{\alpha\beta})^2 \), propagating modes on flat spacetime found by Sezgin+van Nieuwenhuizen 1980 and Kuhfuss+Nitsch 1986. Many exact vacuum and electro-vac solutions were found, interpretation unclear. Cosmological model of Shie-Nester-Yo as example [PRD 78 (2008) 023522] (selection of a consistent model, see Dimakis 1989). Later an extended version was proposed by Chen-Ho-Nester-Wang-Yo (2009):
6. Consistent quadratic Lagrangians for cosmology (selection)

\[ V_{\text{Kop}} \sim \frac{1}{\kappa} R \]
\[ V_{\text{CdeRT}} \sim \frac{1}{\kappa} R + \frac{1}{\varrho} R \wedge^* R \quad \text{[simple } f(R)\text{-grav. in RC-space, de Ritis et al. (1984)]} \]
\[ V_{\text{SNY}} \sim \frac{1}{\kappa} (a_0 R + a_2 V \wedge^* V) + \frac{1}{\varrho} r_6 R \wedge^* R \]
\[ V_{\text{Chen et al.}} \sim \frac{1}{\kappa} (a_0 R + a_2 V \wedge^* V + a_3 A \wedge^* A) + \frac{1}{\varrho} (r_6 R \wedge^* R + r_3 X \wedge^* X) \]
\[ V_{\text{BHN}} \sim \frac{1}{\kappa} \left( a_0 R + b_0 X + a_2 V \wedge^* V + a_3 A \wedge^* A + \sigma_2 A \wedge^* V \right) \]
\[ + \frac{1}{\varrho} (r_6 R \wedge^* R + r_3 X \wedge^* X + \mu_3 R \wedge^* X) \]

Chen et al. found, besides 2 conventional graviton helicities, a massive torsion mode of spin \(0^+\), which doesn’t couple to matter, and a massive torsion mode of spin \(0^-\), which couples to matter of spin \(\frac{1}{2}\). Eventually, BHN (2010) added explicit parity violating pieces linear in the curvature scalar and quadratic in the (axial) vector torsion carrying additional vector modes. The BHN-model may be the most general consistent quadratic model.
7. Field equations for the BHN-Lagrangian in explicit form (PDE’s of 1st order)

Einstein 3-form $G_\alpha := \frac{1}{2} \eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma}$; $\eta_\alpha := *\vartheta_\alpha$, $\eta_{\alpha\beta} := *\vartheta_{\alpha\beta}$, $\eta_{\alpha\beta\gamma} := *\vartheta_{\alpha\beta\gamma}$.

First field eq.:

$$\left( \frac{a_0}{\kappa} - \frac{r_6}{6\varrho} R - \frac{\mu_3}{12\varrho} X \right) G_\alpha + \frac{\lambda_0}{\kappa} \eta_\alpha + \left( \frac{b_0}{\kappa} + \frac{r_3}{6\varrho} X - \frac{\mu_3}{12\varrho} R \right) R_{\alpha\beta} \wedge \vartheta^\beta$$

$$+ \frac{1}{24\varrho} \left( r_3 X^2 - r_6 R^2 - \mu_3 RX \right) \eta_\alpha + \frac{1}{3\kappa} D \left\{ \star \left[ (a_2 V - \sigma_2 A) \wedge \vartheta_\alpha \right] + (a_3 A + \sigma_2 V) \wedge \vartheta_\alpha \right\}$$

$$+ \frac{2a_2}{9\kappa} \left( V_\alpha V^{\beta} - \frac{1}{4} V^2 \delta_\alpha^\beta \right) \eta_\beta + \frac{2a_3}{9\kappa} \left( A_\alpha A^{\beta} - \frac{1}{4} A^2 \delta_\alpha^\beta \right) \eta_\beta$$

$$+ \frac{1}{\kappa} \left( e_\alpha \right) (1) T^\beta_\gamma \left[ \left( a_2 \star 2 T_\beta + a_3 \star 3 T_\beta + \sigma_2 \left( 2 T_\beta + 3 T_\beta \right) \right) \right] = \Sigma_\alpha$$

Second field eq.:

$$\frac{a_0}{\kappa} \left( \star (1) T_{[\alpha} \wedge \vartheta_{\beta]} - \frac{1}{3} V \wedge \eta_{\alpha\beta} + \frac{1}{6} A \wedge \vartheta_{\alpha\beta} \right) + \frac{b_0}{2\kappa} \left( 2 (1) T_{[\alpha} \wedge \vartheta_{\beta]} - \frac{2}{3} V \wedge \vartheta_{\alpha\beta} - \frac{1}{3} A \wedge \eta_{\alpha\beta} \right)$$

$$+ \frac{1}{24\varrho} \left[ \left( 2r_3 dX - \mu_3 dR \right) \wedge \vartheta_{\alpha\beta} - \left( 2r_3 dX - \mu_3 dR \right) \wedge \eta_{\alpha\beta} \right] - \frac{1}{24\varrho} \left( 2r_6 dR + \mu_3 X \right) T^\gamma \wedge \eta_{\alpha\beta\gamma}$$

$$+ \frac{1}{12\varrho} \left( 2r_3 X - \mu_3 R \right) T_{[\alpha} \wedge \vartheta_{\beta]} - \frac{1}{3\kappa} \left[ a_2 V_{[\alpha} \eta_{\beta]} - \sigma_2 A_{[\alpha} \eta_{\beta]} + (a_3 A + \sigma_2 V) \wedge \vartheta_{\alpha\beta} \right] = \tau_{\alpha\beta}$$
8. Friedman ansatz for coframe and connection/torsion

For a homogeneous and isotropic scenario, the *orthonormal coframe* is

\[
\begin{align*}
\vartheta^0 &= dt \\
\vartheta^1 &= \frac{\mathcal{R}(t)dr}{\sqrt{1 - kr^2}} \\
\vartheta^2 &= \mathcal{R}(t)r d\theta \\
\vartheta^3 &= \mathcal{R}(t)rsin \theta d\phi
\end{align*}
\]

with the metric \( g = -\vartheta^0 \otimes \vartheta^0 + \sum_{a=1}^{3} \vartheta^a \otimes \vartheta^a \), where \( \mathcal{R}(t) \) is the expansion factor and \( k \) the curvature index. The *coframe* is denoted by \( e_\alpha \), with \( e_\alpha | \vartheta^\alpha = \delta_\alpha^\beta \). The *Hubble function* is

\[
H(t) := \frac{\mathcal{R}'(t)}{\mathcal{R}(t)}
\]

Homogeneous and isotropic torsion turns out to be

\[
^{(1)}T^\alpha = 0, \quad \mathcal{V} = -3u(t)\vartheta^0, \quad \mathcal{A} = -3v(t)\vartheta^0.
\]

Incidentally, the purely *spatial* part of the torsion \( \mathcal{A} \) corresponds to Cartan’s spiral staircase with a time dependent pitch of the spiral.
9. Accelerating universes

We take for simplicity a spinless perfect fluid as model of matter,

\[ \Sigma_\alpha = T_{\alpha}^{\mu} \eta_\mu \quad \text{with} \quad T_{\alpha}^{\beta} = [\rho(t) + p(t)]U_\alpha U^\beta + p(t)\delta_\alpha^\beta \quad \text{and} \quad \tau_{\alpha\beta} = 0 \]

where \( \rho = \rho(t) \) is the energy density, \( p = p(t) \) the pressure, and \( U_\alpha \) the four-velocity of the fluid, with the normalization \( U^\alpha U_\alpha = -1 \).

By substituting the Friedman and the matter ansätze into the field equations, we find, after some manipulations, six first order ODE’s, see next slide. For the determination of the density we have

\[ \rho'(t) = -3H(t)[\rho(t) + p(t)]. \]

Moreover, we have to choose a constitutive law relating \( \rho \) and \( p \). We took as a first approximation a pressure-free dust, \( p(t) \approx 0 \).
Six first order ODE's for the six observable quantities \( \{R, H, u, v, R, X\} \):

\[
R'(t) = H(t)R(t),
\]

\[
H'(t) = -2H^2(t) - \frac{k}{R^2(t)} + \frac{1}{3a_2} \left\{ \kappa [\rho(t) - 3p(t)] + 4\lambda_0
- \left( a_0 - \frac{a_2}{2} \right) R(t) - (b_0 + \sigma_2) X(t) + \frac{3}{4}(a_2 - 4a_3)v^2(t) \right\},
\]

\[
u'(t) = u^2(t) - 3H(t)u(t) + \frac{1}{3a_2} \left\{ \kappa [\rho(t) - 3p(t)]
+ 4\lambda_0 - a_0 R(t) - (b_0 + \sigma_2) X(t) - 3a_3v^2(t) \right\},
\]

\[
u'(t) = -\frac{1}{3}X(t) - v(t)[3H(t) - 2u(t)],
\]

\[
R'(t) = -2 \left( u(t) + \mu_3 \frac{r_3 - r_6}{B} v(t) \right) R(t) + \frac{\mu_3^2 + 4r_3^2}{B} v(t) X(t)
+ \frac{12g/\kappa}{B} \left[ 2 \left( 2m^+ r_3 + m^\times \mu_3 \right) u(t) + \left( -m^- \mu_3 + 2m^\times r_3 \right) v(t) \right],
\]

\[
X'(t) = -2 \left( u(t) - \mu_3 \frac{r_3 + r_6}{B} v(t) \right) X(t) - \frac{\mu_3^2 + 4r_6^2}{B} v(t) R(t)
+ \frac{12g/\kappa}{B} \left[ \left( 2m^+ \mu_3 - 4m^\times r_6 \right) u(t) + \left( 2m^- r_6 + m^\times \mu_3 \right) v(t) \right],
\]

provided \( B := \mu_3^2 + 4r_3r_6 \neq 0 \) and

\[
m^+ := a_0 - \frac{a_2}{2}, \quad m^- := a_0 - 2a_3, \quad m^\times := b_0 + \sigma_2.
\]
Numerical evaluation of the Chen-Ho-Nester-Wang-Yo model (2009) by using these ODE's with the suitable coupling constants ($T_0 = 1$ is present time):

$\mathcal{R} = a$ expansion factor

$H$

torsion $u = f, v = \chi$

curv. $R, X = E$

energy-dens. $\rho, \rho_\Gamma$

$\mathcal{R}'' = a''$

Courtesy James M. Nester et al. (Chungli)

The BHN-model has been already numerically evaluated by Baekler (to be published). He finds effects reminiscent of those of the Chen et al. model.
10. Discussion

- The PG cosmologies on the basis of the models of SNY, Chen et al., and BHN display realistic features, namely accelerating universes.
- The matter model has to be refined to a spin fluid (see Obukhov and Tresguerres).
- The parity odd pieces and their coupling to matter and antimatter should be worked out.

Soli Deo Gloria